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$\boxed{1\cdot 3-1}$

# Exponential Function

Prove $e^{ix} = \cos x + i\sin x$

We know

$$e^{x} = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \frac{x^5}{\lfloor 5} + \frac{x^6}{\lfloor 6} + \frac{x^7}{\lfloor 7} + \cdots$$

Put $x = ix$

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{\lfloor 2} + \frac{(ix)^3}{\lfloor 3} + \frac{(ix)^4}{\lfloor 4} + \frac{(ix)^5}{\lfloor 5} + \frac{(ix)^6}{\lfloor 6} + \cdots$$

$$= 1 + ix + i^2\frac{x^2}{\lfloor 2} + (i)^3\frac{x^3}{\lfloor 3} + (i)^4\frac{x^4}{\lfloor 4} + (i)^5\frac{x^5}{\lfloor 5} + (i)^6\frac{x^6}{\lfloor 6} + \cdots$$

$$= 1 + ix + i^2\frac{x^2}{\lfloor 2} + i(i^2)\frac{x^3}{\lfloor 3} + (i^2)^2\frac{x^4}{\lfloor 4} + i(i^2)^2\frac{x^5}{\lfloor 5} + i^2(i^2)^2\frac{x^6}{\lfloor 6}$$

$$= 1 + ix - \frac{x^2}{\lfloor 2} - i\frac{x^3}{\lfloor 3} + (1)\frac{x^4}{\lfloor 4} + i(1)\frac{x^5}{\lfloor 5} - (1)\frac{x^6}{\lfloor 6} + \cdots$$

$$= \left(1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots\right) + \left(ix - i\frac{x^3}{\lfloor 3} + i\frac{x^5}{\lfloor 5} - \cdots\right)$$

$$= \left(1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots\right) + i\left(x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \cdots\right)$$

$$= \cos x + i\sin x$$

Note: if $z = x + iy$ in Cartesian form

$z = r(\cos\theta + i\sin\theta)$ in Polar form

$z = re^{i\theta}$ in Exponential form.

Note: $a^{z} = e^{z\ln a}$ if $a \in R$, $a > 0$

$\because \ln a^{z} = \ln e^{z\ln a}$

$z\ln a = z\ln a(\ln e)$

$z\ln a = z\ln a$

## Trigonometric Functions

We know

$e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

$e^{ix} + e^{-ix} = 2\cos x$

$\frac{e^{ix} + e^{-ix}}{2} = \cos x$

Similarly $e^{ix} - e^{-ix} = 2i\sin x$

$\frac{e^{ix} - e^{-ix}}{2i} = \sin x$

$\sec x = \frac{2}{e^{ix} + e^{-ix}}$

$\operatorname{cosec} x = \frac{2i}{e^{ix} - e^{-ix}}$

$\tan x = \frac{\sin x}{\cos x} = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})}$

$\cot x = \frac{i(e^{ix} + e^{-ix})}{e^{ix} - e^{-ix}}$

## Hyperbolic Functions

$\sinh x = \frac{e^{x} - e^{-x}}{2}$  $\operatorname{cosech} x = \frac{2}{e^{x} - e^{-x}}$

$\cosh x = \frac{e^{x} + e^{-x}}{2}$  $\operatorname{sech} x = \frac{2}{e^{x} + e^{-x}}$

$\tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$  $\coth x = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$

## Osborn's Rule

Relation b/w Trig Fns & Hyperbolic fns.

| | |
|---|---|
| $\sin iz = i\sinh z$ | $\sinh iz = i\sin z$ |
| $\cos iz = \cosh z$ | $\cosh iz = \cos z$ |
| $\tan iz = i\tanh z$ | $\tanh iz = i\tan z$ |
| $\cot iz = -i\coth z$ | $\coth iz = -i\cot z$ |
| $\sec iz = \operatorname{sech} z$ | $\operatorname{sech} iz = \sec z$ |
| $\operatorname{cosec} iz = -i\operatorname{cosech} z$ | $\operatorname{cosech} iz = -i\operatorname{cosec} z$ |

Note: To Prove Osborn's Rule just put $iz = z$ & solve.

Prove $\sin iz = i\sinh z$

Proof: $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$

Put $z = iz$

$$\sin iz = \frac{e^{iiz} - e^{-iiz}}{2i} = \frac{e^{i^2 z} - e^{-i^2 z}}{2i} = \frac{e^{(-1)z} - e^{-(-1)z}}{2i}$$

$$= \frac{e^{-z} - e^{z}}{2i} = i\left(\frac{e^{-z} - e^{z}}{2ii}\right) = i\left(\frac{e^{-z} - e^{z}}{-2}\right)$$

$$= i\left(\frac{e^{z} - e^{-z}}{2}\right) = i\sinh z$$

Prove $\cos iz = \cosh z$

$\cos z = \frac{e^{iz} + e^{-iz}}{2}$

Put $z = iz$

$$\cos iz = \frac{e^{iiz} + e^{-iiz}}{2} = \frac{e^{i^2 z} + e^{-i^2 z}}{2}$$

$$= \frac{e^{-z} + e^{z}}{2} = \cosh z$$

Q1 $\boxed{\text{Ex } 1\cdot 3}$

Show that $e^{z}$ is never zero.

Sol: $e^{z}\cdot\frac{1}{e^{z}} = 1$ since the multiplicative inverse of $e^{z}$ exists so $e^{z}$ is never zero

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$$L.H.S = \left(\frac{Z_1}{Z_2}\right)^n = \left(\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}\right)^n$$

$$\Rightarrow \left(\frac{Z_1}{Z_2}\right)^n = \frac{r_1^n}{r_2^n}\left(\frac{(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}\right)^n$$

$$= \frac{r_1^n}{r_2^n}\left(\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1)}{\cos^2\theta_2 + \sin^2\theta_2}\right)^n$$

$$= \frac{r_1^n}{r_2^n}\left(\frac{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)}{1}\right)^n$$

$$= \frac{r_1^n}{r_2^n}\left(\cos n(\theta_1 - \theta_2) + i\sin n(\theta_1 - \theta_2)\right)$$

$$= \frac{r_1^n}{r_2^n}\left(\cos(n\theta_1 - n\theta_2) + i\sin(n\theta_1 - n\theta_2)\right)$$

$$= \frac{r_1^n}{r_2^n}\left((\cos n\theta_1\cos n\theta_2 + \sin n\theta_1\sin n\theta_2) + i(\sin n\theta_1\cos n\theta_2 - \sin n\theta_2\cos n\theta_1)\right)$$

$$= \frac{r_1^n}{r_2^n}\left((\cos n\theta_1\cos n\theta_2 - i^2\sin n\theta_1\sin n\theta_2) + i(\sin n\theta_1\cos n\theta_2 - \sin n\theta_2\cos n\theta_1)\right)$$

$$= \frac{r_1^n}{r_2^n}\left((\cos n\theta_1\cos n\theta_2 - i\sin n\theta_2\cos n\theta_1) + (i\sin n\theta_1\cos n\theta_2 - i^2\sin n\theta_1\sin n\theta_2)\right)$$

$$= \frac{r_1^n}{r_2^n}\left(\cos n\theta_1(\cos n\theta_2 - i\sin n\theta_2) + i\sin n\theta_1(\cos n\theta_2 - i\sin n\theta_2)\right)$$

$$= \frac{r_1^n}{r_2^n}\left((\cos n\theta_1 + i\sin n\theta_1)(\cos n\theta_2 - i\sin n\theta_2)\right)$$

$$= \frac{r_1^n}{r_2^n} \times (\cos\theta_1 + i\sin\theta_1)^n(\cos n\theta_2 + i\sin n\theta_2)^{-1}$$

$$= \frac{r_1^n}{r_2^n}\frac{(\cos\theta_1 + i\sin\theta_1)^n}{(\cos n\theta_2 + i\sin n\theta_2)} = \frac{r_1^n}{r_2^n} \times \frac{(\cos\theta_1 + i\sin\theta_1)^n}{(\cos\theta_2 + i\sin\theta_2)^n} = \frac{Z_1^n}{Z_2^n}$$

$$= R.H.S$$

Q\ 57 1·3+2

(ii) $|e^{iz}|=1$

<u>Proof</u> Since $e^{iz} = \cos z + i \sin z$

$\Rightarrow |e^{iz}| = |\cos z + i \sin z|$

$= \sqrt{\cos^2 z + \sin^2 z} = 1$ = R.H.S.

×————————×

(iv) $e^{z_1} = e^{z_2} \Leftrightarrow z_1 - z_2 = 2k\pi i$, where $k$ is an integer

<u>Proof</u> Suppose that

$e^{z_1} = e^{z_2} \Rightarrow e^{z_1} \cdot e^{-z_2} = e^{z_2} \cdot e^{-z_2}$

$e^{z_1 - z_2} = e^{z_2 - z_2} = e^0 = 1$

Put $z = z_1 - z_2$,

$e^z = 1$, if and only if $z = 2k\pi i$

$k$ is any integer, then

$e^{2k\pi i} = \cos 2k\pi + i \sin 2k\pi$, where $k = 0, \pm 1, \pm 2, ...$

$= 1 + 0i = 1$

( Since $\cos 2k\pi = 1$, $\sin 2k\pi = 0$)

$e^{z_1 - z_2} = 1 \Rightarrow e^{z_1 - z_2} = e^0 = e$

$e^{z_1} \cdot e^{-z_2} = 1 \Rightarrow e^{z_1} = e^{z_2}$

2nd Method

$e^{z_1} = e^{z_2}$

$\Rightarrow \frac{e^{z_1}}{e^{z_2}} = 1$ or

$\Rightarrow e^{z_1 - z_2} = 1$

which is possible only if $z_1 - z_2$ is an integral multiple of $2\pi i$ (as proved in (iii))

$\Rightarrow z_1 - z_2 = 2\pi i K$ where $K$ is integer

Conversely

$z_1 - z_2 = 2\pi i K$

$e^{z_1 - z_2} = e^{2\pi i K}$

$= \cos 2K\pi + i \sin 2K\pi$

$= 1 + 0$

$= 1$

$e^{z_1 - z_2} = 1$

$e^{z_1} = e^{z_2}$

Suppose that

$e^z = 1$, taking $z = x + iy$

$\Rightarrow e^{x+iy} = e^x(\cos y + i \sin y) = 1$

$= e^x \cos y + i e^x \sin y = 1 = 1 + 0i$

$\Rightarrow e^x \cos y = 1$ and $e^x \sin y = 0$

but $e^x \neq 0 \Rightarrow \cos y = 1$ and $\sin y = 0$

(58) $\boxed{1\cdot3-3}$

Q imp

(iii) $e^z = 1 \Leftrightarrow z$ is an integral multiple of $2\pi i$

Proof Let $z = 2\pi i K$ where $K$ is any integer

i.e $K = 0, \pm1, \pm2, \pm3, ---$

Then $e^z = e^{2\pi i K} = \cos 2\pi K + i \sin 2\pi K$ —(E)

$= 1 + 0 = 1$

Since $\cos 2\pi K = 1$ and $\sin 2\pi K = 0$

when $K$ is any integer

Conversely Suppose that $e^z = 1$.

and if $z = x + iy$

$\Rightarrow e^z = e^{x+iy} = e^x e^{iy} = e^x[\cos y + i \sin y]$

$\therefore 1 = e^x \cos y + i e^x \sin y$ $\quad \because e^z = 1$ supposed

Comparing Real & Imaginary Parts

$\Rightarrow 1 = e^x \cos y$ & $0 = e^x \sin y$

$\Rightarrow$ Since $e^x \neq 0 \therefore \cos y \neq 0$ $\quad \Rightarrow \because e^x \neq 0, \therefore \sin y = 0$

$\sin y = 0 \Rightarrow \boxed{y = n\pi}$ where $n$ is any integer

i.e $n = 0, \pm1, \pm2, ---$

Now $\cos y = \cos n\pi = (-1)^n$ where $n = 0, \pm1, \pm2 ---$

$\therefore e^x \cos y = 1$

becomes $e^x(-1)^n = 1$

$\therefore (-1)^n > 0$

Since $e^x > 0$ and product of two +ives or product of two −ives is positive. So $(-1)^n$ is +ive $\because$ product of $e^x$ & $(-1)^n$ is +ive 1 and $e^x > 0$ so $(-1)^n$ must be +ive

$\Rightarrow n$ must be even i.e $\boxed{n = 2K}$ where $K = 0, \pm1, \pm2, ---$

$\therefore \Rightarrow e^x(-1)^{2K} = 1 \Rightarrow e^x \cdot 1 = 1 = e^0 \Rightarrow x = 0$

$\Rightarrow z = 0 + iy = in\pi = i(2K)\pi = 2K\pi i$

where $K = 0, \pm1, \pm2, ---$

$\boxed{1\cdot 3-4.}$

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and it only possible when ~~y is an~~ ~~even integer~~

~~i.e~~ $y = 2k\pi$ and $x = 0$

$\Rightarrow z = 0 + iy = 0 + 2k\pi i$

$\Rightarrow z_1 - z_2 = 2k\pi i$

x————x

(v) Show that $|e^z| = e^x$, when $z = x + iy$

Sol $L.H.S = |e^z| = |e^{x+iy}|$ Since $z = x+iy$ (given)

$= |e^x . e^{iy}| = |e^x||e^{iy}|$

$= |e^x| \, |\cos y + i \sin y|$

$= |e^x| \sqrt{\cos^2 y + \sin^2 y} = |e^x| \cdot 1 = |e^x| = R.H.S$

x————x

(vi) $e^{z_1} . e^{z_2} \ldots\ldots e^{z_n} = e^{z_1 + z_2 + z_3 + \ldots\ldots z_n}$ . where $n = 1, 2, 3 \ldots$

Proof we shall prove it by Induction.

Case-1 Put $n = 2$. then

$e^{z_1} . e^{z_2} = e^{z_1 + z_2}$

Hence c-1 is true for $n=2$

C-2 let it is true for $n = k$.

$e^{z_1} . e^{z_2} . e^{z_3} \ldots\ldots e^{z_k} = e^{z_1 + z_2 + z_3 + \ldots + z_k} \longrightarrow (1)$

Now we have to prove it that it is true for $n = k+1$. for 'x' Eq (1) by $e^{z_{k+1}}$ both sides, we get

2nd Method

$z_1 = (x_1 + iy_1) \quad z_2 = (x_2 + iy_2)$

LHS $e^{x_1 + iy_1} . e^{x_2 + iy_2} . e^{x_3 + iy_3} \ldots\ldots e^{x_n + iy_n}$

$= e^{x_1 + x_2 + \ldots x_n} (\cos y_1 + i \sin y_1) \ldots\ldots (\cos y_n + i \sin y_n)$

$= e^{x_1 + x_2 \ldots x_n} [\cos(y_1 + y_2 \ldots y_n) + i \sin(y_1 + y_2 \ldots y_n)]$

$= e^{x_1 + x_2 \ldots x_n} e^{i(y_1 + y_2 \ldots y_n)}$

$= e^{(x_1 + iy_1) + (x_2 + iy_2) + \ldots (x_n + iy_n)}$

$= e^{z_1 + z_2 + \ldots z_n}$

$$\left(e^{z_1}\cdot e^{z_2}\cdot e^{z_3}\cdots e^{z_k}\right)\cdot e^{z_{k+1}} = e^{z_1+z_2+z_3+\cdots z_k}\cdot e^{z_{k+1}}$$

$$\Rightarrow e^{z_1}\cdot e^{z_2}\cdot e^{z_3}\cdots\cdots e^{z_{k+1}} = e^{z_1+z_2+z_3+\cdots z_k+z_{k+1}}$$

$$= e^{z_1+z_2+\cdots+z_{k+1}}$$

$\Rightarrow$ given statement is true for $n=k+1$. Thus it is true for all +ive integral values of $n$.

x ——— x

**(vii)** $(e^z)^n = e^{nz}$, where $n$ is any integer

Proof let $z = x+iy$, then

$$L.H.S = (e^z)^n = \left(e^{x+iy}\right)^n = \left(e^x\cdot e^{iy}\right)^n = \left[e^x(\cos y + i\sin y)\right]^n$$

$$= e^{nx}\left[\cos ny + i\sin ny\right] \quad (\text{De moivre's Th.})$$

$$= e^{nx}\cdot e^{iny} = e^{nx+iny} = e^{n(x+iy)}$$

$$= e^{nz} = R.H.S.$$

x ——— x

**Q.2 (i)** Prove that $\forall\, z_1, z_2, z_3 \in C$

$$1+\tan^2 z = \sec^2 z$$

L.H.S $1+\tan^2 z$ $\quad\left(\because \tan z = \dfrac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})}\right)$

$$= 1+\left(\frac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})}\right)^2$$

$$= 1+\frac{e^{2iz}+e^{-2iz}-2}{i^2(e^{2iz}+e^{-2iz}+2)} = 1+\frac{2-e^{2iz}-e^{-2iz}}{e^{2iz}+e^{-2iz}+2}$$

$(\because i^2=-1)$

$$= \frac{\cancel{e^{2iz}} + \cancel{e^{-2iz}} + 2 + 2 - \cancel{e^{2iz}} - \cancel{e^{-2iz}}}{e^{2iz} + e^{-2iz} + 2}$$

$$= \frac{4}{e^{2iz} + e^{-2iz} + 2} = \left(\frac{2}{e^{iz} + e^{-iz}}\right)^2 = \sec^2 z$$
$= R.H.S.$

×————×

(ii) $1 + \cot^2 z = \operatorname{cosec}^2 z$



Proof: L.H.S $= 1 + \cot^2 z$

$$= 1 + \left[\frac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}\right]^2 = 1 + \frac{(-1)(e^{2iz} + e^{-2iz} + 2)}{e^{2iz} + e^{-2iz} - 2}$$

(using $i^2 = -1$)

$$= \frac{\cancel{e^{2iz}} + \cancel{e^{-2iz}} - 2 - \cancel{e^{2iz}} - \cancel{e^{-2iz}} - 2}{e^{2iz} + e^{-2iz} - 2}$$

$$= \frac{-4}{e^{2iz} + e^{-2iz} - 2} = \frac{4i^2}{(e^{iz} - e^{-iz})^2} = \left(\frac{2i}{e^{iz} - e^{-iz}}\right)^2$$

$= \operatorname{cosec}^2 z = R.H.S$



×————×

(iii) $\sin(z_1 - z_2) = \sin z_1 \cos z_2 - \cos z_1 \sin z_2$

R.H.S. $\sin z_1 \cos z_2 - \cos z_1 \sin z_2$

$$= \left(\frac{e^{iz_1} - e^{-iz_1}}{2i}\right)\left(\frac{e^{iz_2} + e^{-iz_2}}{2}\right) - \left(\frac{e^{iz_1} + e^{-iz_1}}{2}\right)\left(\frac{e^{iz_2} - e^{-iz_2}}{2i}\right)$$

$$= \frac{e^{iz_1}\cdot e^{iz_2} + e^{-iz_1}\cdot e^{-iz_2} + e^{iz_1}\cdot e^{-iz_2} - e^{-iz_1}\cdot e^{iz_2}}{4i} - \frac{e^{iz_1}\cdot e^{iz_2} - e^{iz_1}\cdot e^{-iz_2} + e^{-iz_1}\cdot e^{iz_2} - e^{-iz_1}\cdot e^{-iz_2}}{4i}$$

$$\frac{\cancel{e^{iz_1+iz_2}} + e^{iz_1-iz_2} - e^{-iz_1+iz_2} - e^{-iz_1-iz_2} - e^{iz_1+iz_2} + e^{iz_1-iz_2} - e^{-iz_1+iz_2} + \cancel{e^{-iz_1-iz_2}}}{4i}$$

$$= \frac{2\left(e^{iz_1-iz_2} - e^{-iz_1+iz_2}\right)}{4i} = \frac{e^{i(z_1-z_2)} - e^{-i(z_1-z_2)}}{2i}$$

$$= \sin(z_1 - z_2) = L.H.S.$$

x ———— x

(iv) $\cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

R.H.S $= \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

$$= \left(\frac{e^{iz_1}+e^{-iz_1}}{2}\right)\left(\frac{e^{iz_2}+e^{-iz_2}}{2}\right) - \left(\frac{e^{iz_1}-e^{-iz_1}}{2i}\right)\left(\frac{e^{iz_2}-e^{-iz_2}}{2i}\right)$$

$$\frac{e^{iz_1}\cdot e^{iz_2} + \cancel{e^{iz_1} e^{-iz_2}} + \cancel{e^{-iz_1}\cdot e^{iz_2}} + e^{-iz_1}\cdot e^{-iz_2}}{4}$$

$$+ \frac{e^{iz_1}e^{iz_2} - \cancel{e^{iz_1}e^{-iz_2}} - \cancel{e^{-iz_1}e^{iz_2}} + e^{-iz_1}e^{-iz_2}}{4}$$

$$= \frac{e^{iz_1}\cdot e^{iz_2} + e^{-iz_1}\cdot e^{-iz_2} + e^{iz_2}e^{iz_1} + e^{-iz_1}\cdot e^{-iz_2}}{4}$$

$$= \frac{2\left(e^{iz_1+iz_2} + e^{-iz_1-iz_2}\right)}{4} = \frac{e^{i(z_1+z_2)} + e^{-i(z_1+z_2)}}{2}$$

$$= \cos(z_1+z_2) = R.H.S.$$

x ———— x

**Q.2(vi)** $\cos 2z = \cos^2 z - \sin^2 z = 2\cos^2 z - 1 = 1 - 2\sin^2 z$

**PROOF** $\cos^2 z - \sin^2 z = \left(\dfrac{e^{iz}+e^{-iz}}{2}\right)^2 - \left(\dfrac{e^{iz}-e^{-iz}}{2i}\right)^2$

$$= \frac{e^{2iz}+e^{-2iz}+2}{4} - \frac{e^{2iz}+e^{-2iz}-2}{4i^2}$$

$$= \frac{e^{2iz}+e^{-2iz}+\cancel{2}}{4} + \frac{e^{2iz}+e^{-2iz}-\cancel{2}}{4}$$

$$= \frac{2\left(e^{2iz}+e^{-2iz}\right)}{4} = \frac{e^{2iz}+e^{-2iz}}{2}$$

$$= \cos 2z = L.H.S$$

$$2\cos^2 z - 1 = 2\left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 - 1$$

$$= \cancel{2}\left(\frac{e^{2iz}+e^{-2iz}+2}{4_{2}}\right) - 1$$

$$= \frac{e^{2iz}+e^{-2iz}+2}{2} - 1 = \frac{e^{2iz}+e^{-2iz}+\cancel{2}-\cancel{2}}{2}$$

$$= \frac{e^{2iz}+e^{-2iz}}{2} = \cos 2z = L.H.S$$

$$1 - 2\sin^2 z = 1 - 2\left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2$$

$$= 1 - 2\left(\frac{e^{2iz}+e^{-2iz}-2}{4i^2}\right)$$

$$= 1 + \frac{e^{2iz}+e^{-2iz}-2}{2} \qquad (\because i^2 = -1)$$

$$= \frac{\cancel{2}+e^{2iz}+e^{-2iz}-\cancel{2}}{2} = \frac{e^{2iz}+e^{-2iz}}{2} = \cos 2z = L.H.S$$

2(vii) $\sin 2z = 2\sin z\cos z$

R.H.S. $= 2\sin z\cos z$

$$= \cancel{2}\left(\frac{e^{iz}-e^{-iz}}{2i}\right)\left(\frac{e^{iz}+e^{-iz}}{\cancel{2}}\right)$$

$$= \frac{e^{2iz}-e^{-2iz}}{2i} = \sin 2z = L.H.S.$$

x———x

2(viii) $\cos z_1 - \cos z_2 = 2\sin\frac{z_1+z_2}{2}\sin\frac{z_2-z_1}{2}$

L.H.S. $\cos z_1 - \cos z_2$

$$= \frac{e^{iz_1}+e^{-iz_1}}{2} - \frac{e^{iz_2}+e^{-iz_2}}{2}$$

$\boxed{\sin z = \frac{e^{iz}-e^{-iz}}{2i}}$

$$= \left(e^{iz_1}+e^{-iz_1}-e^{iz_2}-e^{-iz_2}\right)/2 \longrightarrow (1)$$

R.H.S. $2\sin\frac{z_1+z_2}{2}\sin\frac{z_2-z_1}{2}$

$$= \cancel{2}\left(\frac{e^{i\left(\frac{z_1+z_2}{2}\right)}-e^{-i\left(\frac{z_1+z_2}{2}\right)}}{\cancel{2}i}\right)\left(\frac{e^{i\left(\frac{z_2-z_1}{2}\right)}-e^{-i\left(\frac{z_2-z_1}{2}\right)}}{2i}\right)$$

$$= \frac{1}{2i^2}\left\{e^{\frac{iz_1+iz_2}{2}}\cdot e^{\frac{iz_2-iz_1}{2}} - e^{\frac{iz_1+iz_2}{2}}\cdot e^{\frac{-iz_2+iz_1}{2}} - e^{\frac{-iz_1-iz_2}{2}}\,e^{\frac{iz_2-iz_1}{2}} + e^{\frac{-iz_1-iz_2}{2}}\cdot e^{\frac{-iz_2+iz_1}{2}}\right\}$$

$$= -\frac{1}{2}\left\{e^{\frac{2iz_2}{2}} - e^{\frac{2iz_1}{2}} - e^{\frac{-2iz_1}{2}} + e^{\frac{-2iz_2}{2}}\right\}$$

$$= \frac{e^{iz_1}+e^{-iz_1}-e^{iz_2}-e^{-iz_2}}{2} \longrightarrow (2)$$

from (1) and (2)

L.H.S. = R.H.S.

$\boxed{1\cdot 3-10}$ 66

**Q.2 (ix)** $\sin z_1 + \sin z_2 = 2\sin\frac{z_1+z_2}{2}\cos\frac{z_1-z_2}{2}$

$$L.H.S. = \sin z_1 + \sin z_2$$

$$= \frac{e^{iz_1} - e^{-iz_1}}{2i} + \frac{e^{iz_2} - e^{-iz_2}}{2i}$$

$$= \frac{e^{iz_1} - e^{-iz_1} + e^{iz_2} - e^{-iz_2}}{2i} \longrightarrow (1)$$

$$R.H.S. = 2\sin\frac{z_1+z_2}{2}\cos\frac{z_1-z_2}{2}$$

$$= \cancel{2}\left(\frac{e^{\frac{i(z_1+z_2)}{2}} - e^{-i\left(\frac{z_1+z_2}{2}\right)}}{2i}\right)\left(\frac{e^{i\left(\frac{z_1-z_2}{2}\right)} + e^{-i\left(\frac{z_1-z_2}{2}\right)}}{\cancel{2}}\right)$$

$$= \frac{1}{2i}\left[\begin{array}{l} e^{\frac{iz_1+iz_2}{2}}\cdot e^{\frac{iz_1-iz_2}{2}} + e^{\frac{iz_1+iz_2}{2}}\cdot e^{\frac{-iz_1+iz_2}{2}} \\ - e^{\frac{-iz_1-iz_2}{2}}\cdot e^{\frac{iz_1-iz_2}{2}} - e^{\frac{-iz_1-iz_2}{2}}\cdot e^{\frac{-iz_1+iz_2}{2}} \end{array}\right]$$

$$= \frac{1}{2i}\left[e^{\frac{2iz_1}{2}} + e^{\frac{2iz_2}{2}} - e^{\frac{-2iz_2}{2}} - e^{\frac{-2iz_1}{2}}\right]$$

$$= \frac{1}{2i}\left(e^{iz_1} - e^{-iz_1} + e^{iz_2} - e^{-iz_2}\right) \longrightarrow (2)$$

from (1) and (2) $\Rightarrow$ L.H.S. = R.H.S.

**Q.2 (x)** $\sin 3z = 3\sin z - 4\sin^3 z$

L.H.S. $3\sin z - 4\sin^3 z$

$$= 3\left(\frac{e^{iz} - e^{-iz}}{2i}\right) - 4\left(\frac{e^{iz} - e^{-iz}}{2i}\right)^3$$

$$= 3\left(\frac{e^{iz} - e^{-iz}}{2i}\right) - 4\left(\frac{e^{3iz} - 3e^{2iz}e^{-iz} + 3e^{iz}e^{-2iz} - e^{-3iz}}{8i^3}\right)$$

$$= \frac{3e^{iz} - 3e^{-iz}}{2i} - 4\left(\frac{e^{3iz} - 3e^{iz} + 3e^{-iz} - e^{-3iz}}{-4(2i)}\right)$$

$$= \frac{3e^{iz} - 3e^{-iz} + e^{3iz} - 3e^{iz} + 3e^{-iz} - e^{-3iz}}{2i}$$

$$= \frac{e^{3iz} - e^{-3iz}}{2i} = \sin 3z = L.H.S.$$

**Q.2(xi)** $\tan(z_1+z_2) = \dfrac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2}$

**Sol** If $z_1, z_2$ are complex numbers, then

$\sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

and $\cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

$\Rightarrow L.H.S. = \tan(z_1+z_2)$

$$= \frac{\sin(z_1+z_2)}{\cos(z_1+z_2)} \text{ , putting values we get}$$

$$= \frac{\sin z_1 \cos z_2 + \cos z_1 \sin z_2}{\cos z_1 \cos z_2 - \sin z_1 \sin z_2}$$

$\because$ each term of nume and denome- by $\cos z_1 \cdot \cos z_2$, we get

$$= \frac{\frac{\sin z_1 \cos z_2}{\cos z_1 \cos z_2} + \frac{\cos z_1 \sin z_2}{\cos z_1 \cos z_2}}{1 - \frac{\sin z_1}{\cos z_1}\frac{\sin z_2}{\cos z_2}}$$

$$= \frac{\frac{\sin z_1}{\cos z_1} + \frac{\sin z_2}{\cos z_2}}{1 - \frac{\sin z_1}{\cos z_1}\cdot\frac{\sin z_2}{\cos z_2}}$$

$$= \frac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2} = R.H.S$$

**Q·2(xii)** $\tan(z_1 - z_2) = \dfrac{\tan z_1 - \tan z_2}{1 + \tan z_1 \tan z_2}$

L.H.S $\tan(z_1 - z_2)$

$$= \frac{\sin(z_1 - z_2)}{\cos(z_1 - z_2)} \longrightarrow (i)$$

we know that if $z_1$ and $z_2$ are any complex numbers then

$$\sin(z_1 - z_2) = \sin z_1 \cos z_2 - \cos z_1 \sin z_2$$

and $\cos(z_1 - z_2) = \cos z_1 \cos z_2 + \sin z_1 \sin z_2$

putting these values in (i) we get

$$\tan(z_1 - z_2) = \frac{\sin z_1 \cos z_2 - \cos z_1 \sin z_2}{\cos z_1 \cos z_2 + \sin z_1 \sin z_2}$$

$\div$ nume- and denome- by $\cos z_1 \cos z_2$

we get

$$\tan(z_1 - z_2) = \frac{\frac{\sin z_1 \cos z_2}{\cos z_1 \cos z_2} - \frac{\cos z_1 \sin z_2}{\cos z_1 \cos z_2}}{1 + \frac{\sin z_1}{\cos z_1} \cdot \frac{\sin z_2}{\cos z_2}}$$

$$= \frac{\frac{\sin z_1}{\cos z_1} - \frac{\sin z_2}{\cos z_2}}{1 + \frac{\sin z_1}{\cos z_1} \cdot \frac{\sin z_2}{\cos z_2}}$$

$$= \frac{\tan z_1 - \tan z_2}{1 + \tan z_1 \cdot \tan z_2} = R.H.S.$$

X————————X

**1.3 (i)** Show that

$$\overline{\sin z} = \sin \bar{z}$$

L.H.S $\overline{\sin z}$

let $z = x + iy$, then

$$\sin z = \sin(x+iy)$$
$$= \sin x \cos(iy) + \cos x \sin(iy)$$
$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

$\because \cos iz = \cosh z$ and $\sin iz = i \sinh z$

$$\Rightarrow \overline{\sin z} = \overline{\sin x \cosh y + i \cos x \sinh y}$$
$$= \sin x \cosh y - i \cos x \sinh y \longrightarrow (1)$$

and $\sin \bar{z} = \sin(\overline{x+iy}) = \sin(x-iy)$
$$= \sin x \cos(iy) - \sin(iy) \cos x$$
$$= \sin x \cosh y - i \sinh y \cos x \longrightarrow (2)$$

from (1) and (2)
$$\overline{\sin z} = \sin \bar{z}$$

x ——————— x

**Q.3(ii)** $\overline{\cos z} = \cos \bar{z}$

$\because \cos iz = \cosh z$ and $\sin iz = i \sinh z$

L.H.S $= \overline{\cos z}$

let $z = x+iy$, then
$$\cos z = \cos(x+iy) = \cos x \cos iy - \sin x \sin iy$$
$$= \cos x \cosh y - i \sin x \sinh y$$

$$\Rightarrow \overline{\cos z} = \cos x \cosh y + i \sin x \sinh y \longrightarrow (1)$$

and $\cos \bar{z} = \cos(\overline{x+iy}) = \cos(x-iy)$
$$= \cos x \cos iy + \sin x \sin iy$$
$$= \cos x \cosh y + i \sin x \sinh y \longrightarrow (2)$$

from (1) and (2)
$$\overline{\cos z} = \cos \bar{z}$$

x ——————— x



**Q.3(iii)** $\overline{\tan z} = \tan \bar{z}$

Sol L.H.S $= \overline{\tan z}$

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let $z = x+iy$

then $\tan z = \tan(x+iy)$

$$= \frac{\tan x + \tan iy}{1-\tan x \tan iy} \qquad \because \tan iz = i\tanh z$$

$$\tan z = \frac{\tan x + i\tanh y}{1 - i\tan x \tanh y}$$

$$\Rightarrow \overline{\tan z} = \overline{\left(\frac{\tan x + i\tanh y}{1 - i\tan x \tanh y}\right)} \qquad \boxed{\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}}$$

$$= \frac{\overline{\tan x + i\tanh y}}{\overline{1 - i\tan x \tanh y}}$$

$$= \frac{\tan x - i\tanh y}{1 + i\tan x \tanh y} \qquad \cancel{\tan} \bar{x}$$

$$= \frac{\tan x - \tan iy}{1+\tan x \tan iy} = \tan(x-iy) = \tan \bar{z}$$

$$= R.H.S.$$

×————×

**Q.3(iv)** $\sin(-z) = -\sin z$

L.H.S $\quad \sin(-z) = \dfrac{e^{i(-z)} - e^{-i(-z)}}{2i}$

$$= \frac{e^{-iz} - e^{iz}}{2i} = -\left(\frac{e^{iz} - e^{-iz}}{2i}\right)$$

$$= -\sin z = R.H.S$$

×————×

**Q.3(v)** $\cos(-z) = \cos z$

L.H.S $\quad \cos(-z) = \dfrac{e^{i(-z)} + e^{-i(-z)}}{2}$

$$= \frac{e^{-iz} + e^{+iz}}{2} = \frac{e^{iz} + e^{-iz}}{2} = \cos z = R.H.S$$

×————×

**Q.3(vi)** $\tan(-z) = -\tan z$

L.H.S. $\tan(-z) = \dfrac{e^{i(-z)} - e^{-i(-z)}}{i\left(e^{i(-z)} + e^{-i(-z)}\right)} = \dfrac{e^{-iz} - e^{iz}}{i\left(e^{-iz} + e^{iz}\right)}$



$$= -\left[\frac{e^{iz} - e^{-iz}}{i\left(e^{iz} + e^{-iz}\right)}\right] = -\tan z = R.H.S.$$

×————×

**Q.3(vii)** $\sinh(-z) = -\sinh z$

L.H.S. $\sinh(-z) = \dfrac{e^{-z} - e^{-(-z)}}{2} = \dfrac{e^{-z} - e^{+z}}{2}$

$$= -\left(\frac{e^{z} - e^{-z}}{2}\right) = -\sinh z = R.H.S.$$

×————×

**Q.3(viii)** $\cosh(-z) = \cosh z$

L.H.S. $= \cosh(-z) = \dfrac{e^{-z} + e^{-(-z)}}{2} = \dfrac{e^{z} + e^{z}}{2} = \dfrac{e^{z} + e^{-z}}{2} = \cosh z$
$= R.H.S$

×————×

**Q.3(ix)** $\tanh(-z) = -\tanh z$

L.H.S $= \tanh(-z) = \dfrac{e^{-z} - e^{-(-z)}}{e^{-z} + e^{-(-z)}} = \dfrac{e^{-z} - e^{z}}{e^{-z} + e^{z}}$

$$= -\left(\frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}\right) = -\tanh z = R.H.S.$$

×————×

**Q.3(x)** $\overline{\tanh z} = \tanh \bar{z}$

Sol We know that $i\tanh z = \tan(iz)$

Ex # $\boxed{1.3-16}$

Q4 (i) $\cosh^2 z - \sinh^2 z = 1$ To Prove

$\cos^2 z + \sin^2 z = 1$ by putting $z = iz$

$\cos^2 iz + \sin^2 iz = 1$ $\because \cos iz = \cosh z$

$\cosh^2 z + (i\sinh z)^2 = 1$ $\because \sin iz = i\sinh z$

$\cosh^2 z - \sinh^2 z = 1$ $\because i^2 = -1$



(ii) To Prove $\operatorname{sech}^2 z = 1 - \tanh^2 z$

We know

$\sec^2 z = 1 + \tan^2 z$ by putting $z = iz$

$\sec^2 iz = 1 + \tan^2 iz$ $\because \sec iz = \operatorname{sech} z$

$\operatorname{sech}^2 z = 1 + (i\tanh z)^2$ $\because \tan iz = i\tanh z$

$\operatorname{sech}^2 z = 1 - \tanh^2 z$ $\because i^2 = -1$

(iii) To Prove $\operatorname{cosech}^2 z = \coth^2 z - 1$



We know

$1 + \cot^2 z = \operatorname{cosec}^2 z$ by putting $z = iz$

$1 + \cot^2 iz = \operatorname{cosec}^2 iz$ $\because \cot iz = -i\coth z$

$1 + (-i\coth z)^2 = (-i\operatorname{cosech} z)^2$ $\because \operatorname{cosec} iz = -i\operatorname{cosech} z$

$1 - \coth^2 z = -\operatorname{cosech}^2 z$

$\operatorname{cosech}^2 z = \coth^2 z - 1$

(iv) $\cosh 2z = \cosh^2 z + \sinh^2 z = 2\cosh^2 z - 1 = 1 + 2\sinh^2 z$

We know

$\cos 2z = \cos^2 z - \sin^2 z$

$\cos 2iz = (\cos iz)^2 - (\sin iz)^2$

$= \cosh^2 z - (i\sinh z)^2$

$\cosh 2z = \cosh^2 z + \sinh^2 z$ proved

$= \cosh^2 z + (\cosh^2 z - 1)$

$= 2\cosh^2 z - 1$ proved

$= 2(1 + \sinh^2 z) - 1$

$= 2 + 2\sinh^2 z - 1$

$= 1 + 2\sinh^2 z$ proved

OR $\cosh 2z = 2\cosh^2 z - 1$

We know $\cos 2z = 2\cos^2 z - 1$

$\cos 2iz = 2(\cos iz)^2 - 1$

$\cosh 2z = 2\cosh^2 z - 1$ proved

OR $\cosh 2z = 1 + 2\sinh^2 z$

We know

$\cos 2z = 1 - 2\sin^2 z$

$\cos 2(iz) = 1 - 2(\sin iz)^2$

$\cosh 2z = 1 - 2(i\sinh z)^2$

$\cosh 2z = 1 + 2\sinh^2 z.$ proved

Q4(ix) $\tanh(z_1 \pm z_2) = \dfrac{\tanh z_1 \pm \tanh z_2}{1 \pm \tanh z_1 \tanh z_2}$

Sol We know

$$\tan(z_1 \pm z_2) = \frac{\tan z_1 \pm \tan z_2}{1 \mp \tan z_1 \tan z_2}$$

$$\tan(iz_1 \pm iz_2) = \frac{\tan iz_1 \pm \tan iz_2}{1 \mp \tan iz_1 \tan iz_2}$$ by putting $z_1 = iz_1$, $z_2 = iz_2$

$$\tan i(z_1 + z_2) = \frac{i\tanh z_1 \pm i\tanh z_2}{1 \mp (i)^2 \tanh z_1 \tanh z_2}$$ $(\because \tan i z_1 = i\tanh z_1)$

$$i\tanh(z_1 + z_2) = i\left(\frac{\tanh z_1 \pm \tanh z_2}{1 \pm \tanh z_1 \tanh z_2}\right)$$

$$\tanh(z_1 + z_2) = \frac{\cancel{i}(\tanh z_1 \pm \tanh z_2)}{\cancel{i}(1 \pm \tanh z_1 \tanh z_2)}$$ Proved

Q4(x) $\tanh 3z = \dfrac{3\tanh z + \tanh^3 z}{1 + 3\tanh^2 z}$

Sol We know

$$\tan 3z = \frac{3\tan z - \tan^3 z}{1 - 3\tan^2 z}$$

$$\tan 3iz = \frac{3\tan iz - \tan^3 iz}{1 - 3\tan^2 iz}$$

$$i\tanh 3z = \frac{3i\tanh z - (i\tanh z)^3}{1 - 3ii\tanh^2 z}$$

$$i\tanh 3z = \frac{i(3\tanh z + \tanh^3 z)}{1 + 3\tanh^2 z}$$

$$\tanh 3z = \frac{\cancel{i}(3\tanh z + \tanh^3 z)}{\cancel{i}(1 + 3\tanh^2 z)}$$ Proved

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Q4 (v) To Prove $\sinh 2z = 2\sinh z \cosh z$

We know

$\sin 2z = 2\sin z \cos z$

$\sin 2iz = 2\sin iz \cos iz$

$i\sinh 2z = 2(i\sinh z)(\cosh z)$

$\sinh 2z = 2\sinh z \cosh z$ proved

(vi) To Prove $\sinh 3z = 3\sinh z + 4\sinh^3 z$

We know

$\sin 3z = 3\sin z - 4\sin^3 z$

$\sin 3iz = 3\sin iz - 4\sin^3 iz$

$i\sinh 3z = 3i\sinh z - 4(i\sinh z)^3$

$i\sinh 3z = 3i\sinh z + 4i\sinh^3 z$

$\sinh 3z = \frac{\cancel{i}}{\cancel{i}}(3\sinh z + 4\sinh^3 z)$

proved

vii) To Prove $\cosh 3z = 4\cosh^3 z - 3\cosh z$

We know

$\cos 3z = 4\cos^3 z - 3\cos z$

$\cos 3iz = 4\cos^3 iz - 3\cos iz$

$\cosh 3z = 4\cosh^3 z - 3\cosh z$ proved

viii) To Prove $\sinh(z_1 - z_2) = \sinh z_1 \cosh z_2 - \cosh z_1 \sinh z_2$

We know

$\sin(z_1 - z_2) = \sin z_1 \cos z_2 - \cos z_1 \sin z_2$

$\sin(iz_1 - iz_2) = \sin iz_1 \cos iz_2 - \cos iz_1 \sin iz_2$

$\sin i(z_1 - z_2) = i\sinh z_1 \cosh z_2 - \cosh z_1 (i\sinh z_2)$

$i\sinh(z_1 - z_2) = i(\sinh z_1 \cosh z_2 - \cosh z_1 \sinh z_2)$

$\sinh(z_1 - z_2) = \sinh z_1 \cosh z_2 - \cosh z_1 \sinh z_2$

proved

1·3 – 19    85

Q5 (i) If $z = x + iy$, Prove that
$\sin z = \sin x \cosh y + i \cos x \sinh y$

Sol $\sin z = \sin(x+iy)$
$= \sin x \cos iy + \cos x \sin iy$
$= \sin x \cosh y + i \cos x \sinh y$ Proved

ii) Prove that $\tan z = \dfrac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}$

Nr & Dr by $2\cos(x-iy)$

$\tan z = \tan(x+iy)$
$= \dfrac{\sin(x+iy)}{\cos(x+iy)}$
$= \dfrac{2\sin(x+iy)\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)}$
$= \dfrac{\sin(x+\cancel{iy}+x-\cancel{iy}) + \sin(\cancel{x}+iy-\cancel{x}+iy)}{\cos(x+\cancel{iy}+x-\cancel{iy}) + \cos(\cancel{x}+iy-\cancel{x}+iy)}$
$= \dfrac{\sin 2x + \sin 2iy}{\cos 2x + \cos 2iy}$
$= \dfrac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}$ Proved

2nd Method

$\tan z = \tan(x+iy) = \dfrac{\sin(x+iy)}{\cos(x+iy)} = \dfrac{\sin x \cos iy + \cos x \sin iy}{\cos x \cos iy - \sin x \sin iy}$

$= \dfrac{\sin x \cosh y + i \cos x \sinh y}{\cos x \cosh y - i \sin x \sinh y}$

$= \dfrac{(\sin x \cosh y + i \cos x \sinh y)}{(\cos x \cosh y - i \sin x \sinh y)} \times \dfrac{(\cos x \cosh y + i \sin x \sinh y)}{(\cos x \cosh y + i \sin x \sinh y)}$

$= \dfrac{\sin x \cos x \cosh^2 y - \cos x \sin x \sinh^2 y + i(\sin^2 x \cosh y \sinh y + \cos^2 x \sinh y \cosh y)}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}$

$= \dfrac{\sin x \cos x (\cosh^2 y - \sinh^2 y) + i \cosh y \sinh y (\sin^2 x + \cos^2 x)}{\cos^2 x (1 + \sinh^2 y) + \sin^2 x \sinh^2 y}$

(Nr & Dr by 2)

$= \dfrac{\sin x \cos x \cdot 1 + i \cosh y \sinh y}{\cos^2 x + \sinh^2 y (\cos^2 x + \sin^2 x)} = \dfrac{2\sin x \cos x + i\, 2\cosh y \sinh y}{2\cos^2 x + 2\sinh^2 y}$

$\because \cos 2x = 2\cos^2 x - 1$
$\because \cosh 2y = 2\sinh^2 y + 1$
$\sinh 2y = 2\cosh y \sinh y$

$= \dfrac{\sin 2x + i \sinh 2y}{(2\cos^2 x - 1) + (2\sinh^2 y + 1)} = \dfrac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}$ proved

(2+) Ex 1.4    Ex 1.4 (i)

Q11 Prove that
$\sec(x+iy) = \dfrac{2(\cos x \cosh y + i \sin x \sinh y)}{\cos 2x + \cosh 2y}$

Sol $\sec(x+iy)$    Nr & Dr by $2\cos(x-iy)$
$= \dfrac{1}{\cos(x+iy)} \times \dfrac{2\cos(x-iy)}{2\cos(x-iy)}$
$= \dfrac{2(\cos x \cos iy + \sin x \sin iy)}{2\cos(x+iy)\cos(x-iy)}$
$= \dfrac{2(\cos x \cosh y + i \sin x \sinh y)}{\cos(x+\cancel{iy}+x-\cancel{iy}) + \cos(\cancel{x}+iy-\cancel{x}+iy)}$
$= \dfrac{2(\cos x \cosh y + i \sin x \sinh y)}{\cos 2x + \cos 2iy}$
$= \dfrac{2(\cos x \cosh y + i \sin x \sinh y)}{\cos 2x + \cosh 2y}$ proved

Q6 If $\sin(A+iB) = x+iy$ then show that,

(i) $\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1$ (ii) $\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1$

Sol
$\sin(A+iB) = x+iy$
$\sin A \cos iB + \cos A \sin iB = x+iy$
$\Rightarrow \sin A \cosh B + i\cos A \sinh B = x+iy$
Equating Real & Imaginary Parts
$\sin A \cosh B = x$ —(i) & $\cos A \sinh B = y$ —(ii)
$\frac{x}{\sin A} = \cosh B$ $\frac{y}{\cos A} = \sinh B$
Squaring
$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = \cosh^2 B - \sinh^2 B$
$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1$ proved

from (i) $\frac{x}{\cosh B} = \sin A$
from (ii) $\frac{y}{\sinh B} = \cos A$
Squaring + adding
$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = \sin^2 A + \cos^2 A$
$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1$
proved

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Q7 (i) If $\tan(\alpha+i\beta) = x+iy$, then
Show that $x^2+y^2+2x\cot 2\alpha = 1$

Sol $\tan(\alpha+i\beta) = x+iy$
$\alpha+i\beta = \tan^{-1}(x+iy)$ —(i)
Conjugate $\alpha - i\beta = \tan^{-1}(x-iy)$ —(ii)
Add (i)&(ii) To eliminate $\beta$ $\alpha+\cancel{i\beta}+\alpha-\cancel{i\beta} = \tan^{-1}(x+iy)+\tan^{-1}(x-iy)$
$2\alpha = \tan^{-1}\left(\frac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)}\right)$
$2\alpha = \tan^{-1}\left[\frac{2x}{1-(x^2+y^2)}\right]$
$\tan 2\alpha = \frac{2x}{1-x^2-y^2}$
Reciprocal
$\cot 2\alpha = \frac{1-x^2-y^2}{2x}$
$2x\cot 2\alpha = 1-x^2-y^2$
$x^2+y^2+2x\cot 2\alpha = 1$
proved.

(ii) If $\tan(\alpha+i\beta) = x+iy$ then show that $x^2+y^2-2y\coth 2\beta = -1$

Sol $\tan(\alpha+i\beta) = x+iy$
$\alpha+i\beta = \tan^{-1}(x+iy)$ —(i)
$\alpha-i\beta = \tan^{-1}(x-iy)$ —(ii)
(To eliminate $\alpha$) Subtracting
$\alpha+i\beta-(\alpha-i\beta) = \tan^{-1}(x+iy)-\tan^{-1}(x-iy)$
$2i\beta = \tan^{-1}\left[\frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)}\right]$
$\tan 2i\beta = \frac{2iy}{1+x^2+y^2}$
$i\tanh 2\beta = \frac{2iy}{1+x^2+y^2}$
$\tanh 2\beta = \frac{2y}{1+x^2+y^2}$
$\coth 2\beta = \frac{1+x^2+y^2}{2y}$
$2y\coth 2\beta = 1+x^2+y^2$
$-1 = x^2+y^2-2y\coth 2\beta$
proved

Note $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

1·3–21

Q8 If $\sin(\theta+\phi i) = \cos\alpha + i\sin\alpha$, prove that $\cos^2\theta = \pm\sin\alpha$

Sol $\sin(\theta+\phi i) = \cos\alpha + i\sin\alpha$

$\sin\theta\cos(i\phi) + \cos\theta\sin(i\phi) = \cos\alpha + i\sin\alpha$

$\sin\theta\cosh\phi + i\cos\theta\sinh\phi = \cos\alpha + i\sin\alpha$

Equating real & imaginary parts

$\sin\theta\cosh\phi = \cos\alpha$ & $\cos\theta\sinh\phi = \sin\alpha$

$\cosh\phi = \dfrac{\cos\alpha}{\sin\theta}$ $\qquad$ $\sinh\phi = \dfrac{\sin\alpha}{\cos\theta}$

squaring & subtracting

$$\cosh^2\phi - \sinh^2\phi = \frac{\cos^2\alpha}{\sin^2\theta} - \frac{\sin^2\alpha}{\cos^2\theta}$$

$$1 = \frac{\cos^2\alpha\cos^2\theta - \sin^2\alpha\sin^2\theta}{\sin^2\theta\cos^2\theta}$$

$$\sin^2\theta\cos^2\theta = \cos^2\alpha\cos^2\theta - \sin^2\alpha\sin^2\theta$$

$$(1-\cos^2\theta)\cos^2\theta = (1-\sin^2\alpha)\cos^2\theta - \sin^2\alpha(1-\cos^2\theta)$$

$$\cos^2\theta - \cos^4\theta = \cos^2\theta - \cancel{\sin^2\alpha\cos^2\theta} - \sin^2\alpha + \cancel{\sin^2\alpha\cos^2\theta}$$

$$\cancel{\cos^2\theta} - \cancel{\cos^2\theta} + \sin^2\alpha = \cos^4\theta$$

$$\pm\sin\alpha = \cos^2\theta \quad \text{Proved.}$$

Q10 Prove that $\sinh\left(\frac{x}{2}\right) = \sqrt{\dfrac{\cosh x - 1}{2}}$ if $x \geq 0$

$= -\sqrt{\dfrac{\cosh x-1}{2}}$ if $x < 0$

Sol RHS $\sqrt{\dfrac{\cosh x - 1}{2}}$

$$= \sqrt{\frac{\left(\frac{e^x+e^{-x}}{2}\right)-1}{2}} = \sqrt{\frac{e^x+e^{-x}-2}{4}}$$

$$= \sqrt{\frac{\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^2}{4}} = \pm\left(\frac{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}{2}\right) = \pm\sinh\left(\frac{x}{2}\right)$$

$\Rightarrow \sqrt{\dfrac{\cosh x-1}{2}} = \sinh\left(\frac{x}{2}\right)$ for $x \geq 0$

& $\sqrt{\dfrac{\cosh x-1}{2}} = -\sinh\left(\frac{x}{2}\right)$ for $x<0$

$\Rightarrow \sinh\left(\frac{x}{2}\right) = \sqrt{\dfrac{\cosh x-1}{2}}$ for $x \geq 0$

$\Rightarrow \sinh\left(\frac{x}{2}\right) = -\sqrt{\dfrac{\cosh x-1}{2}}$ for $x<0$

proved

2nd Method

$$\cosh x = 1 + 2\sinh^2\frac{x}{2}$$

$$\cosh x - 1 = 2\sinh^2\frac{x}{2}$$

$$\frac{\cosh x-1}{2} = \sinh^2\frac{x}{2}$$

$$\pm\sqrt{\frac{\cosh x-1}{2}} = \sinh\frac{x}{2}$$

1·3-22

**Q9. (i)** If $\tan(\theta+\phi i) = \tan\alpha + i\sec\alpha$,

Prove that $e^{2\phi} = \pm\cot\frac{\alpha}{2}$

**Sol:** $\tan(\theta+\phi i) = \tan\alpha + i\sec\alpha$

$\theta+\phi i = \tan^{-1}(\tan\alpha + i\sec\alpha)$ — (i)

$\theta-\phi i = \tan^{-1}(\tan\alpha - i\sec\alpha)$ — (ii)

Subtracting:

$2i\phi = \tan^{-1}(\tan\alpha + i\sec\alpha) - \tan^{-1}(\tan\alpha - i\sec\alpha)$

$2i\phi = \tan^{-1}\left[\dfrac{(\tan\alpha + i\sec\alpha) - (\tan\alpha - i\sec\alpha)}{1+(\tan\alpha+i\sec\alpha)(\tan\alpha - i\sec\alpha)}\right]$

$\tan 2i\phi = \dfrac{2i\sec\alpha}{1+\tan^2\alpha+\sec^2\alpha}$

$i\tanh 2\phi = \dfrac{2i\sec\alpha}{\sec^2\alpha+\sec^2\alpha}$

$\tanh 2\phi = \dfrac{\cancel{2}\sec\alpha}{\cancel{2}\sec^2\alpha}$

$= \dfrac{1}{\sec\alpha}$

$\tanh 2\phi = \cos\alpha$

$\dfrac{e^{2\phi}-e^{-2\phi}}{e^{2\phi}+e^{-2\phi}} = \dfrac{\cos\alpha}{1}$

Componendo & Dividendo $\left(\because \frac{a}{b} = \frac{a+b}{a-b}\right)$

$\dfrac{(e^{2\phi}-e^{-2\phi})+(e^{2\phi}+e^{-2\phi})}{(e^{2\phi}-e^{-2\phi})-(e^{2\phi}+e^{-2\phi})} = \dfrac{\cos\alpha+1}{\cos\alpha-1}$

$\dfrac{\cancel{2}e^{2\phi}}{-\cancel{2}e^{-2\phi}} = \dfrac{\cos\alpha+1}{-(1-\cos\alpha)}$

$e^{4\phi} = \dfrac{2\cos^2\alpha/2}{2\sin^2\frac{\alpha}{2}}$

$e^{4\phi} = \cot^2\frac{\alpha}{2}$

$\boxed{e^{2\phi} = \pm\cot\frac{\alpha}{2}}$

**(ii)** If $\tan(\theta+\phi i) = \tan\alpha + i\sec\alpha$

Prove that $2\theta = n\pi + \frac{\pi}{2} + \alpha$

**Sol:** Add (i)+(ii)

$2\theta = \tan^{-1}(\tan\alpha+i\sec\alpha)+\tan^{-1}(\tan\alpha-i\sec\alpha)$

$= \tan^{-1}\left[\dfrac{(\tan\alpha+i\sec\alpha)+(\tan\alpha-i\sec\alpha)}{1-(\tan\alpha+i\sec\alpha)(\tan\alpha-i\sec\alpha)}\right]$

$\tan 2\theta = \dfrac{2\tan\alpha}{1-(\tan^2\alpha+\sec^2\alpha)}$

$= \dfrac{2\tan\alpha}{1-\tan^2\alpha-\sec^2\alpha}$

$= \dfrac{2\tan\alpha}{1-\tan^2\alpha-(1+\tan^2\alpha)}$

$= \dfrac{2\tan\alpha}{-2\tan^2\alpha}$

$= -\dfrac{1}{\tan\alpha}$

$\tan 2\theta = -\cot\alpha$

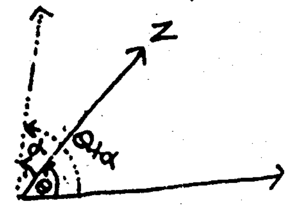
$= \tan\left(\frac{\pi}{2}+\alpha\right)$

$\tan 2\theta = \tan\left(n\pi+\frac{\pi}{2}+\alpha\right)$

$2\theta = n\pi+\frac{\pi}{2}+\alpha$

proved.

Q11 Show that multiplication of a vector $z$ by $e^{i\alpha}$, where $\alpha$ is a real number rotates the vector $z$ counterclockwise through an angle '$\alpha$'.




Sol $z = r(\cos\theta + i\sin\theta)$

$= r\, e^{i\theta}$

$z\, e^{i\alpha} = r\, e^{i\theta} e^{i\alpha}$

$= r\, e^{i(\theta+\alpha)}$

$z\, e^{i\alpha} = r\left(\cos(\theta+\alpha) + i\sin(\theta+\alpha)\right)$

Thus vector $z$ is rotated through an angle $\alpha$ counterclockwise.

Q12 Show that $2+i = \sqrt{5}\, e^{i\tan^{-1}\left(\frac{1}{2}\right)}$

$z = 2+i \quad \Rightarrow r = |z| = \sqrt{2^2+1^2} = \sqrt{5}$

$\cos\theta = \dfrac{x}{r} = \dfrac{2}{\sqrt{5}}$

$\sin\theta = \dfrac{y}{r} = \dfrac{1}{\sqrt{5}}$

$\tan\theta = \dfrac{1}{2} \quad \Rightarrow \quad \theta = \tan^{-1}\dfrac{1}{2}$



In polar form $z = r(\cos\theta + i\sin\theta) = r\, e^{i\theta}$

$2+i = \sqrt{5}\, e^{i\tan^{-1}\frac{1}{2}}$

(ii) $z = -3-4i \quad \Rightarrow \quad r = |z| = \sqrt{9+16} = \sqrt{25} = 5$

$\cos\theta = \dfrac{x}{r} = \dfrac{-3}{5}$

$\sin\theta = \dfrac{y}{r} = \dfrac{-4}{5}$

$\tan\theta = \dfrac{-4}{-3} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{4}{3}\right)$

$\theta = \pi + \tan^{-1}\left(\frac{4}{3}\right) \quad \because$ 3rd Quad

In polar form $z = r\, e^{i\theta}$

$= 5\, e^{i\left(\pi + \tan^{-1}\frac{4}{3}\right)}$ Ans.

Logarithmic Function:-

Let $z \in$ where $z \neq 0$

& $w \in$ such that $e^{w} = z \Rightarrow w = \ln z$, 'w' is Natural logarithm of z.

A particular value of 'w' satisfying the eq $e^{w} = z$ is given by $\ln|z| + i\arg$

$e^{\ln|z| + i\arg z} = e^{\ln|z|} e^{i\arg z} = |z| e^{i\theta} = r(\cos\theta + i\sin\theta) = z$

This particular value of $\omega$ is called Principal logarithm of z denoted by Log z.

$\boxed{\text{Log } z = \ln|z| + i\arg z} = \boxed{\ln\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x} \quad \text{if } z = x+iy}$

The general value of $w$ satisfying the eq $e^{w} = z$ is given by $\ln|z| + i\arg z + 2n\pi i$

$e^{\ln|z| + i\arg z + 2n\pi i} = e^{\ln|z|} \cdot e^{i\arg z} \cdot e^{2n\pi i} = |z| e^{i\theta} e^{2\pi n i}$

$= r e^{i\theta} (\cos 2\pi n + i\sin 2\pi n)$

$= r e^{i\theta} (1+0)$

$= r(\cos\theta + i\sin\theta)$

$= z$

Ex 1.4

$\text{Log } z = \ln|z| + i\arg z$

when $z = x+iy$

$\text{Log}(x+iy) = \ln\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x}$

Q1(i) Prove that $\text{Log } i = \frac{\pi i}{2}$

$\text{Log } i = \text{Log}(0+1i)$

$= \ln\sqrt{0^2+1^2} + i\tan^{-1}\left(\frac{1}{0}\right)$

$= \ln 1 + i\frac{\pi}{2}$

$= 0 + \frac{i\pi}{2}$

$= \frac{i\pi}{2}$ Ans

$x+ive, y+ive$

So Ist Quad

$\therefore \theta$

$\theta = \tan^{-1}(\infty) = \frac{\pi}{2}$

ii) $\text{Log}(-5) = \ln 5 + i\pi$ To Prove

$\text{Log}(-5) = \ln\sqrt{(-5)^2+0^2} + i\tan^{-1}\left(\frac{0}{-5}\right)$

$= \ln\sqrt{25} + i\tan^{-1}(0)$

$= \ln 5 + i\pi$

$x$-ive $y$+ive

So IInd Q

$\therefore \pi - \theta$

$\theta = \tan^{-1}(0) = 0$

Principal arg $z = \pi - \theta = \pi - 0$

$= \pi$

$\boxed{1\cdot 4-2}$

Q1(iii) $Log(-1+i) = \frac{1}{2}\ln 2 + \frac{3}{4}\pi i$ To Prove

$$Log(-1+i) = \ln\sqrt{1+1} + i\tan^{-1}\left(\frac{1}{-1}\right)$$
$$= \ln\sqrt{2} + i\tan^{-1}(-1)$$
$$= \ln 2^{\frac{1}{2}} + i\tan^{-1}(-1)$$
$$= \frac{1}{2}\ln 2 + i\,\frac{3\pi}{4}$$

x -ive y +ive
So IInd Quad.
$\therefore \pi-\theta$
$\theta = \tan^{-1}(1) = \frac{\pi}{4}$
Principal arg z $= \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

---

iv) $Log(1+i) = \frac{1}{2}\ln 2 + \frac{\pi}{4} i$ To Prove

$$Log(1+i) = \ln\sqrt{1^2+1^2} + i\tan^{-1}\left(\frac{1}{1}\right)$$
$$= \ln\sqrt{2} + i\tan^{-1} 1$$
$$= \frac{1}{2}\ln 2 + i\,\frac{\pi}{4}$$

x +ive, y +ive
Ist Quad $\therefore \theta$
$\theta = \tan^{-1}(1) = \frac{\pi}{4}$

---

v) $Log\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = -\frac{2}{3}\pi i$ To Prove

$$Log\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = \ln\sqrt{\frac{1}{4}+\frac{3}{4}} + i\tan^{-1}\left(\frac{-\sqrt{3}/2}{-1/2}\right)$$
$$= \ln\sqrt{\frac{4}{4}} + i\tan^{-1}(\sqrt{3})$$
$$= \ln 1 + i\left(-\frac{2\pi}{3}\right)$$
$$= 0 - \frac{2}{3}\pi i$$

x -ive, y -ive
3rd Quad.
$\therefore \theta - \pi$
$\theta = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$
$\therefore \theta - \pi = \frac{\pi}{3} - \pi$
$= -\frac{2\pi}{3}$

---

vi) $Log(1-i) = \frac{1}{2}\ln 2 - \frac{\pi}{4} i$ To Prove

$$Log(1-i) = \ln\sqrt{1^2+(-1)^2} + i\tan^{-1}\left(\frac{-1}{1}\right)$$
$$= \ln\sqrt{2} + i\tan^{-1}(-1)$$
$$= \frac{1}{2}\ln 2 + i\left(-\frac{\pi}{4}\right)$$
$$= \frac{1}{2}\ln 2 - \frac{\pi}{4} i$$

x +ive, y -ive
So 4th Quad
$\therefore -\theta$
$\theta = \tan^{-1}(1) = \frac{\pi}{4}$
Principal arg z $= -\theta = -\frac{\pi}{4}$

---

$\boxed{1.4-3}$



Q2 i) To Prove $\coth^{-1}z = \frac{1}{2}\log\left(\frac{z+1}{z-1}\right)$

Let $\coth^{-1}z = w \Rightarrow z = \coth w$

$$\frac{z+1}{z-1} = \frac{\coth w + 1}{\coth w - 1}$$

$$= \frac{\frac{e^{w}+e^{-w}}{e^{w}-e^{-w}}+1}{\frac{e^{w}+e^{-w}}{e^{w}-e^{-w}}-1}$$

$$= \frac{e^{w}\cancel{+e^{-w}}+e^{w}\cancel{-e^{-w}}}{\cancel{e^{w}}+e^{-w}\cancel{-e^{w}}+e^{-w}} \quad \text{LCM}$$

$$= \frac{\cancel{2}e^{w}}{\cancel{2}e^{-w}}$$

$$= e^{w}e^{w}$$

$$= e^{2w}$$

$$\log\left(\frac{z+1}{z-1}\right) = 2w$$

$$\frac{1}{2}\log\left(\frac{z+1}{z-1}\right) = w = \coth^{-1}z$$

To Prove $\operatorname{sech}^{-1}z = \log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$

Let $\operatorname{sech}^{-1}z = w \Rightarrow z = \operatorname{sech} w$

So, $$\frac{1+\sqrt{1-z^2}}{z} = \frac{1+\sqrt{1-\operatorname{sech}^2 w}}{\operatorname{sech} w}$$

$(\because \operatorname{sech}^2 z + \tanh^2 z = 1)$

$$= \frac{1+\tanh w}{\operatorname{sech} w}$$

$$= \frac{1}{\operatorname{sech} w} + \left(\frac{\sinh w}{\cancel{\cosh w}}\right)\cdot\cancel{\cosh w}$$

$$= \cosh w + \sinh w$$

$$= \frac{e^{w}+e^{-w}}{2} + \frac{e^{w}-e^{-w}}{2}$$

$$= \frac{\cancel{2}e^{w}}{\cancel{2}} \quad \text{LCM}$$

$$\log\left(\frac{1+\sqrt{1-z^2}}{z}\right) = w$$

$$\boxed{\log\left(\frac{1+\sqrt{1-z^2}}{z}\right) = \operatorname{sech}^{-1}z} \text{ proved}$$

⑥ To Prove $\operatorname{cosech}^{-1}z = \log\left(\frac{1+\sqrt{1+z^2}}{z}\right)$

Let $\operatorname{cosech}^{-1}z = w \Rightarrow z = \operatorname{cosech} w$

So $$\frac{1+\sqrt{1+z^2}}{z} = \frac{1+\sqrt{1+\operatorname{cosech}^2 w}}{\operatorname{cosech} w}$$

$\because \coth^2 z - \operatorname{cosech}^2 z = 1$

$$= \frac{1+\sqrt{\coth^2 w}}{\operatorname{cosech} w}$$

$$= \frac{1+\coth w}{\operatorname{cosech} w}$$

$$= \frac{1}{\operatorname{cosech} w} + \frac{\coth w}{\operatorname{cosech} w}$$

$$= \sinh w + \frac{\cosh w}{\cancel{\sinh w}}\cdot\frac{\cancel{\sinh w}}{1}$$

$$= \frac{e^{w}-e^{-w}}{2} + \frac{e^{w}+e^{-w}}{2}$$

$$= \frac{\cancel{2}e^{w}}{\cancel{2}}$$

$$\log\left(\frac{1+\sqrt{1+z^2}}{z}\right) = w$$

$$\boxed{\log\left(\frac{1+\sqrt{1+z^2}}{z}\right) = \operatorname{cosech}^{-1}z} \text{ proved}$$



Hyperbolic Identities

$$\begin{cases} \cosh^2 z - \sinh^2 z = 1 \\ \operatorname{sech}^2 z + \tanh^2 z = 1 \\ \coth^2 z - \operatorname{cosech}^2 z = 1 \end{cases}$$

Trig Identities

$$\begin{cases} \cos^2\theta + \sin^2\theta = 1 \\ \sec^2\theta - \tan^2\theta = 1 \\ \operatorname{cosec}^2\theta - \cot^2\theta = 1 \end{cases}$$

## Inverse Hyperbolic Functions :-

**To Prove** $\sinh^{-1} z = \log\left(z+\sqrt{1+z^2}\right)$

Let $\sinh^{-1} z = \omega \Rightarrow z = \sinh\omega$

So $z+\sqrt{1+z^2} = \sinh\omega + \sqrt{1+\sinh^2\omega}$

$= \sinh\omega + \sqrt{\cosh^2\omega}$

$= \sinh\omega + \cosh\omega$

$= \frac{e^{\omega}-e^{-\omega}}{2} + \frac{e^{\omega}+e^{-\omega}}{2}$

$= \frac{\cancel{2}e^{\omega}}{\cancel{2}}$ LCM

$\log\left(z+\sqrt{z^2+1}\right) = \omega$

$\boxed{\log\left(z+\sqrt{z^2+1}\right) = \sinh^{-1} z}$ proved

**To Prove** $\cosh^{-1} z = \log\left(z+\sqrt{z^2-1}\right)$

Let $\cosh^{-1} z = \omega \Rightarrow z = \cosh\omega$

So $z+\sqrt{z^2-1} = \cosh\omega + \sqrt{\cosh^2\omega - 1}$

$= \cosh\omega + \sqrt{\sinh^2\omega}$

$= \cosh\omega + \sinh\omega$

$= \frac{e^{\omega}+e^{-\omega}}{2} + \frac{e^{\omega}-e^{-\omega}}{2}$

$= \frac{\cancel{2}e^{\omega}}{\cancel{2}}$ LCM

$\log\left(z+\sqrt{z^2-1}\right) = \omega$

$\boxed{\log\left(z+\sqrt{z^2-1}\right) = \cosh^{-1} z}$

**IHF**

1) $\sinh^{-1} z = \log\left(z+\sqrt{1+z^2}\right)$
2) $\cosh^{-1} z = \log\left(z+\sqrt{z^2-1}\right)$
3) $\tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$
4) $\coth^{-1} z = \frac{1}{2}\log\left(\frac{z+1}{z-1}\right)$
5) $\operatorname{sech}^{-1} z = \log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$
6) $\operatorname{cosech}^{-1} z = \log\left(\frac{1+\sqrt{z^2+1}}{z}\right)$

3) **To Prove** $\tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$

Let $\tanh^{-1} z = \omega \Rightarrow z = \tanh\omega$

So $\frac{1+z}{1-z} = \frac{1+\tanh\omega}{1-\tanh\omega}$

$= \frac{1+\left(\frac{e^{\omega}-e^{-\omega}}{e^{\omega}+e^{-\omega}}\right)}{1-\left(\frac{e^{\omega}-e^{-\omega}}{e^{\omega}+e^{-\omega}}\right)}$

$= \frac{e^{\omega}+\cancel{e^{-\omega}}+e^{\omega}-\cancel{e^{-\omega}}}{\cancel{e^{\omega}}+e^{-\omega}-\cancel{e^{\omega}}+e^{-\omega}}$ LCM

$= \frac{\cancel{2}e^{\omega}}{\cancel{2}e^{-\omega}}$

$= e^{\omega}\cdot e^{\omega}$

$\frac{1+z}{1-z} = e^{2\omega}$

$\log\left(\frac{1+z}{1-z}\right) = 2\omega$

$\frac{1}{2}\log\left(\frac{1+z}{1-z}\right) = \omega = \tanh^{-1} z$

1.4 - 4

## Inverse Trigonometric Functions :-

$$\sin^{-1} z = \frac{1}{i}\log(iz+\sqrt{1-z^2})$$
$$\cos^{-1} z = \frac{1}{i}\log(z+\sqrt{z^2-1})$$
$$\tan^{-1} z = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right)$$
$$\sec^{-1} z = \frac{1}{i}\log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$$
$$\operatorname{cosec}^{-1} z = \frac{1}{i}\log\left(\frac{i+\sqrt{z^2-1}}{z}\right)$$
$$\cot^{-1} z = \frac{1}{2i}\log\left(\frac{z+i}{z-i}\right)$$

1) To Prove $\sin^{-1} z = \frac{1}{i}\log(iz+\sqrt{1-z^2})$

Let $\sin^{-1} z = w \Rightarrow z = \sin w$

So $iz+\sqrt{1-z^2} = i\sin w + \sqrt{1-\sin^2 w}$
$= i\sin w + \sqrt{\cos^2 w}$
$= i\sin w + \cos w$
$= e^{iw}$

$\log(iz+\sqrt{1-z^2}) = iw$

$\frac{1}{i}\log(iz+\sqrt{1-z^2}) = w$

$$\boxed{\frac{1}{i}\log(iz+\sqrt{1-z^2}) = \sin^{-1} z}$$

Q 3(i) 2) To Prove $\cos^{-1} z = \frac{1}{i}\log(z+\sqrt{z^2-1})$

Let $\cos^{-1} z = w \Rightarrow z = \cos w$

So $z+\sqrt{z^2-1} = \cos w + \sqrt{\cos^2 w - 1}$
$= \cos w + \sqrt{-(1-\cos^2 w)}$
$= \cos w + \sqrt{-\sin^2 w}$
$= \cos w + i\sin w$

$z+\sqrt{z^2-1} = e^{iw}$

$\log(z+\sqrt{z^2-1}) = iw$

$\frac{1}{i}\log(z+\sqrt{z^2-1}) = w$

$$\boxed{\frac{1}{i}\log(z+\sqrt{z^2-1}) = \cos^{-1} z}$$

3) To Prove $\tan^{-1} z = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right)$

Let $\tan^{-1} z = w \Rightarrow z = \tan w$

So $\frac{1+iz}{1-iz} = \frac{1+i\tan w}{1-i\tan w}$
$= \frac{1+i\frac{\sin w}{\cos w}}{1-i\frac{\sin w}{\cos w}}$

LCM $= \frac{\cos w + i\sin w}{\cos w - i\sin w}$
$= \frac{e^{iw}}{e^{-iw}}$
$= e^{iw}e^{iw}$

$\frac{1+iz}{1-iz} = e^{2iw}$

$\log\left(\frac{1+iz}{1-iz}\right) = 2iw$

$\frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right) = w$

$$\boxed{\frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right) = \tan^{-1} z}$$

Q3 (ii) To Prove $\cot^{-1} z = \frac{1}{2i}\log\left(\frac{z+i}{z-i}\right)$

Let $\cot^{-1} z = \omega \Rightarrow z = \cot\omega$

So $\frac{z+i}{z-i} = \frac{\cot\omega + i}{\cot\omega - i}$

$$= \frac{\frac{\cos\omega}{\sin\omega} + i}{\frac{\cos\omega}{\sin\omega} - i}$$

$$= \frac{\cos\omega + i\sin\omega}{\cos\omega - i\sin\omega}$$

$$= \frac{e^{i\omega}}{e^{-i\omega}} = e^{i\omega}e^{i\omega}$$

$$= e^{2i\omega}$$

$\log\left(\frac{z+i}{z-i}\right) = 2i\omega$

$\frac{1}{2i}\log\left(\frac{z+i}{z-i}\right) = \omega$

$$\boxed{\frac{1}{2i}\log\left(\frac{z+i}{z-i}\right) = \cot^{-1} z}$$ Proved

(6) To Prove $\operatorname{Cosec}^{-1} z = \frac{1}{i}\log\left(\frac{i+\sqrt{z^2-1}}{z}\right)$

Let $\operatorname{Cosec}^{-1} z = \omega \Rightarrow z = \operatorname{Cosec}\omega$

So $\frac{i+\sqrt{z^2-1}}{z} = \frac{i+\sqrt{\operatorname{Cosec}^2\omega - 1}}{\operatorname{Cosec}\omega}$

$$= \frac{i+\sqrt{\cot^2\omega}}{\operatorname{Cosec}\omega}$$

$$= \frac{i+\cot\omega}{\operatorname{Cosec}\omega}$$

$$= \frac{i+\frac{\cos\omega}{\sin\omega}}{\frac{1}{\sin\omega}}$$ Lcm

$$= \frac{i\sin\omega + \cos\omega}{1}$$

$$= e^{i\omega}$$

$\log\frac{i+\sqrt{z^2-1}}{z} = i\omega$

$\frac{1}{i}\log\left[\frac{i+\sqrt{z^2-1}}{z}\right] = \omega$

$$\boxed{\frac{1}{i}\log\left(\frac{i+\sqrt{z^2-1}}{z}\right) = \operatorname{Cosec}^{-1} z}$$ Proved

Q3 (iv) To Prove $\operatorname{Sec}^{-1} z = \frac{1}{i}\log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$

Let $\operatorname{Sec}^{-1} z = \omega \Rightarrow z = \operatorname{Sec}\omega$

So $\frac{1+\sqrt{1-z^2}}{z} = \frac{1+\sqrt{1-\operatorname{Sec}^2\omega}}{\operatorname{Sec}\omega}$

$$= \frac{1+\sqrt{-\tan^2\omega}}{\operatorname{Sec}\omega} \quad \left(\because \operatorname{Sec}^2\omega - \tan^2\omega = 1,\ -\tan^2\omega = 1 - \operatorname{Sec}^2\omega\right)$$

$$= \frac{1+i\tan\omega}{\operatorname{Sec}\omega}$$

$$= \left(1 + i\frac{\sin\omega}{\cos\omega}\right)\cdot\frac{\cos\omega}{1}$$

$$= \frac{\cos\omega + i\sin\omega}{\cos\omega}\cdot\frac{\cos\omega}{1}$$

$\frac{1+\sqrt{1-z^2}}{z} = e^{i\omega}$

$\log\left(\frac{1+\sqrt{1-z^2}}{z}\right) = i\omega \Rightarrow \omega = \frac{1}{i}\log\left(\frac{1+\sqrt{1-z^2}}{z}\right)^2$

$\operatorname{Sec}^{-1} z = \frac{1}{i}\log\left(\frac{1+\sqrt{1-z^2}}{z}\right)^2$ proved

**Complex Power** If $z$ and $w$ are any two complex numbers s.t $z \neq 0$

$$\boxed{z^{w} = e^{w \operatorname{Log} z}}$$

then we define

Q1 Prove that $i^{i} = e^{-\pi/2}$

LHS $i^{i} = e^{i \operatorname{Log} i}$

$= e^{i\{\ln\sqrt{0^2+1^2} + i\tan^{-1}(\frac{1}{0})\}}$ $\because$ QI

$= e^{i(0 + i\frac{\pi}{2})}$

$i^{i} = e^{-\frac{\pi}{2}}$ proved

ii) Prove that $(-1)^{i} = e^{-\pi}$

LHS $(-1)^{i} = e^{i \operatorname{Log}(-1)}$

$= e^{i\{\ln\sqrt{(-1)^2+0^2} + i\tan^{-1}(0)\}}$

$= e^{i[0 + i\pi]}$ $\because$ QII

$(-1)^{i} = e^{-\pi}$ proved

iii) Prove that $(-i)^{-i} = e^{-\frac{\pi}{2}}$

LHS $(-i)^{-i} = e^{-i \operatorname{Log}(-i)}$

$= e^{-i\{\ln\sqrt{0^2+(-1)^2} + i\tan^{-1}(\frac{-1}{0})\}}$

$= e^{-i\{0 + i\tan^{-1}(-\infty)\}}$ $\because$ QIV

$= e^{-i\{0 + i(-\frac{\pi}{2})\}}$

$= e^{-\frac{\pi}{2}}$ proved

iv) Prove that $a^{i} = \cos(\ln a) + i\sin(\ln a)$

LHS $a^{i} = e^{i \operatorname{Log} a}$

$= e^{i[\ln\sqrt{a^2+0^2} + i\tan^{-1}(\frac{0}{a})]}$

$= e^{i[\ln a + i(0)]}$ $\because$ $I^{th}$ Q

$= e^{i\ln a}$

$a^{i} = \cos\ln a + i\sin\ln a$ proved

**Q5** Prove that $\tanh^{-1} z = \sinh^{-1}\left(\frac{z}{\sqrt{1-z^2}}\right)$

Sol Let $\tanh^{-1} z = w \Rightarrow z = \tanh w$

So $\frac{z}{\sqrt{1-z^2}} = \frac{\tanh w}{\sqrt{1-\tanh^2 w}}$

$= \frac{\tanh w}{\sqrt{\operatorname{sech}^2 w}}$

$= \frac{\tanh w}{\operatorname{sech} w}$

$= \frac{\sinh w}{\cosh w} \cdot \frac{\cosh w}{1}$

$= \sinh w$

$\sinh^{-1}\left(\frac{z}{\sqrt{1-z^2}}\right) = w$

$\sinh^{-1}\left(\frac{z}{\sqrt{1-z^2}}\right) = \tanh^{-1} z$

**Q6** Show that if $z = x+iy$ then $\operatorname{Log}\left(\frac{z}{\bar{z}}\right) = 2i\tan^{-1}\left(\frac{y}{x}\right)$

Sol $z = x+iy$

$\bar{z} = x-iy$

$\operatorname{Log}\left(\frac{z}{\bar{z}}\right) = \operatorname{Log} z - \operatorname{Log}\bar{z}$

$= \operatorname{Log}(x+iy) - \operatorname{Log}(x-iy)$

$= [\ln\sqrt{x^2+y^2} + i\tan^{-1}(\frac{y}{x})] - [\ln\sqrt{x^2+y^2} + i\tan^{-1}(\frac{-y}{x})]$

$= \ln\sqrt{x^2+y^2} + i\tan^{-1}(\frac{y}{x}) - \ln\sqrt{x^2+y^2} - (i\tan^{-1}\frac{y}{x})$

$\operatorname{Log}\frac{z}{\bar{z}} = 2i\tan^{-1}\frac{y}{x}$ proved

$\boxed{1.4-7}$

Q) If $a^{\alpha+i\beta} = (x+iy)^{p+iq}$ then Prove that

i) $\alpha = \frac{1}{2}\log_a(x^2+y^2) - \tan^{-1}(\frac{y}{x})\log_a e$

ii) $\log_a(x^2+y^2) = 2(\alpha \ldots \beta q)$

Sol

$a^{\alpha+i\beta} = (x+iy)^{p+iq}$

$e^{(\alpha+i\beta)\operatorname{Log} a} = e^{(p+iq)\operatorname{Log}(x+iy)}$



$(\alpha+i\beta)\operatorname{Log} a = (p+iq)\operatorname{Log}(x+iy)$

$(\alpha+i\beta)\left[\ln\sqrt{a^2} + i\tan^{-1}\frac{0}{a}\right] = (p+iq)\left[\ln\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x}\right]$

$(\alpha+i\beta)\left[\ln a + i\cdot 0\right] = (p+iq)\left[\frac{1}{2}\ln(x^2+y^2) + i\tan^{-1}\frac{y}{x}\right]$

$\alpha\ln a + i\beta\ln a = \left[\frac{p}{2}\ln(x^2+y^2) - q\tan^{-1}\frac{y}{x}\right] + i\left[\frac{q}{2}\ln(x^2+y^2) + p\tan^{-1}\frac{y}{x}\right]$ — (1)

Equating Real & Imaginary Parts

$\alpha\ln a = \frac{p}{2}\ln(x^2+y^2) - q\tan^{-1}\frac{y}{x}$

$\beta\ln a = \frac{q}{2}\ln(x^2+y^2) + p\tan^{-1}\frac{y}{x}$

$\beta = \frac{q\ln(x^2+y^2)}{\ldots} + \frac{p\tan^{-1}\frac{y}{x}}{\ldots}$

Q8 If $\log \sin(x+iy) = u+iv$ show that

i) $\cosh 2y = \cos 2x + 2e^{2u}$

ii) $e^{2y} = \dfrac{\cos(x-v)}{\cos(x+v)}$

Sol $\log \sin(x+iy) = u+iv$ given

$$\sin(x+iy) = e^{u+iv}$$

$$\sin x \cos iy + \cos x \sin iy = e^{u}\cdot e^{iv}$$

$$\sin x \cosh y + i\cos x \sinh y = e^{u}(\cos v + i\sin v)$$

Equating Real & Imaginary parts.

$$\sin x \cosh y = e^{u}\cos v \quad \text{—(i)}$$

$$\cos x \sinh y = e^{u}\sin v \quad \text{—(ii)}$$

Squaring & adding (i) & (ii) (To change into double angle as required)

$\because 2\cosh^2\theta - 1 = \cosh 2\theta$
$1 + 2\sinh^2\theta = \cosh 2\theta$

$$\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y = e^{2u}(\cos^2 v + \sin^2 v)$$

$$\left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cosh 2y}{2}\right) + \left(\frac{1+\cos 2x}{2}\right)\left(\frac{\cosh 2y - 1}{2}\right) = e^{2u}$$

$$\frac{1-\cos 2x+\cosh 2y-\cos 2x\cosh 2y}{4} + \frac{\cosh 2y+\cos 2x\cosh 2y-1-\cos 2x}{4} = e^{2u}$$

$$\frac{\cancel{1}-\cancel{1}-2\cos 2x+2\cosh 2y-\cancel{\cos 2x\cosh 2y}+\cancel{\cos 2x\cosh 2y}}{4} = e^{2u}$$

$$2(\cosh 2y - \cos 2x) = 4e^{2u}$$

$$\cosh 2y - \cos 2x = 2e^{2u}$$

$$\boxed{\cosh 2y = \cos 2x + 2e^{2u}} \text{ proved}$$

Comp & Div. $\left(\frac{a}{b} = \frac{a+b}{a-b}\right)$

Divide eq (i) by (ii)

$$\frac{\sin x \cosh y}{\cos x \sinh y} = \frac{\cos v}{\sin v}$$

$$\frac{\cosh y}{\sinh y} = \frac{\cos v \cos x}{\sin v \sin x}$$

$$\frac{e^{y}+e^{-y}}{e^{y}-e^{-y}} = \frac{\cos v\cos x}{\sin v \sin x}$$

Now apply Componendo & Dividendo

$$\frac{e^{y}+\cancel{e^{-y}}+e^{y}-\cancel{e^{-y}}}{\cancel{e^{y}}+e^{-y}-\cancel{e^{y}}+e^{-y}} = \frac{\cos v\cos x + \sin v \sin x}{\cos v \cos x - \sin v \sin x}$$

$$\frac{\cancel{2}e^{y}}{\cancel{2}e^{-y}} = \frac{\cos(x-v)}{\cos(x+v)}$$

$$e^{y}e^{y} = \frac{\cos(x-v)}{\cos(x+v)}$$

$$e^{2y} = \frac{\cos(x-v)}{\cos(x+v)} \text{ proved}$$

⑨ Show that $\text{Log}(1+\cos\theta+i\sin\theta) = \ln\left(2\cos\frac{\theta}{2}\right)+i\frac{\theta}{2}$

LHS $\text{Log}(1+\cos\theta+i\sin\theta)$

$$= \ln\sqrt{(1+\cos\theta)^2+\sin^2\theta} + i\tan^{-1}\left(\frac{\sin\theta}{1+\cos\theta}\right)$$

$$= \ln\sqrt{1+\cos^2\theta+2\cos\theta+\sin^2\theta} + i\tan^{-1}\left(\frac{\cancel{2}\sin\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}}{\cancel{2}\cos^{\cancel{2}}\frac{\theta}{2}}\right)$$

$$= \ln\sqrt{1+1+2\cos\theta} + i\tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$= \ln\sqrt{2(1+\cos\theta)} + i\frac{\theta}{2}$$

$$= \ln\sqrt{2\left(2\cos^2\frac{\theta}{2}\right)} + i\frac{\theta}{2}$$

$$= \ln 2\cos\frac{\theta}{2} + i\frac{\theta}{2} \quad \text{proved}$$

Imp ⑩ Prove that $\tan^{-1}\left(\frac{x+iy}{x-iy}\right) = \frac{\pi}{4}+\frac{i}{2}\ln\left(\frac{x+y}{x-y}\right)$

Let $\alpha+i\beta = \tan^{-1}\left(\frac{x+iy}{x-iy}\right)$ —①

$\alpha-i\beta = \tan^{-1}\left(\frac{x-iy}{x+iy}\right)$ —②

Add ① & ②

$$2\alpha = \tan^{-1}\left[\frac{\frac{x+iy}{x-iy}+\frac{x-iy}{x+iy}}{1-\cancel{\left(\frac{x+iy}{x-iy}\right)}\cancel{\left(\frac{x-iy}{x+iy}\right)}}\right]$$

$$2\alpha = \tan^{-1}(\infty) = \frac{\pi}{2}$$

$$\boxed{\alpha = \frac{\pi}{4}}$$

Subtracting ② from ①

$$2i\beta = \tan^{-1}\left[\frac{\frac{x+iy}{x-iy}-\frac{x-iy}{x+iy}}{1+\cancel{\frac{x+iy}{x-iy}}\,\cancel{\frac{x-iy}{x+iy}}}\right]$$

$$\tan 2i\beta = \frac{(x+iy)^2-(x-iy)^2}{2(x^2+y^2)}$$

$$i\tanh 2\beta = \frac{\cancel{x^2}\cancel{-y^2}+2ixy-\cancel{x^2}+\cancel{y^2}+2ixy}{2(x^2+y^2)}$$

$$i\tanh 2\beta = \frac{2ixy}{x^2+y^2}$$

$$\tanh 2\beta = \frac{2xy}{x^2+y^2}$$

$$\frac{e^{2\beta}-e^{-2\beta}}{e^{2\beta}+e^{-2\beta}} = \frac{2xy}{x^2+y^2}$$

By Comp & Dividendo

$$\Rightarrow \frac{e^{2\beta}\cancel{-e^{-2\beta}}+e^{2\beta}\cancel{+e^{-2\beta}}}{\cancel{e^{2\beta}}-e^{-2\beta}-\cancel{e^{2\beta}}-e^{-2\beta}} = \frac{2xy+x^2+y^2}{2xy-x^2-y^2}$$

$$\frac{\cancel{2}e^{2\beta}}{\cancel{-2}e^{-2\beta}} = \frac{(x+y)^2}{-(x-y)^2}$$

$$e^{2\beta}e^{2\beta} = \left(\frac{x+y}{x-y}\right)^2$$

$$e^{4\beta} = \left(\frac{x+y}{x-y}\right)^2$$

Take Square Root

$$e^{2\beta} = \frac{x+y}{x-y}$$

Take log

$$2\beta = \ln\left(\frac{x+y}{x-y}\right)$$

$$\beta = \frac{1}{2}\ln\left(\frac{x+y}{x-y}\right)$$

Hence

$$\tan^{-1}\left(\frac{x+iy}{x-iy}\right) = \frac{\pi}{4}+\frac{i}{2}\ln\left(\frac{x+y}{x-y}\right)$$

proved.

$\boxed{1.4 - 10}$

See Pag # 85 Ex 1.4(i) 7P

Q# To Prove $\cos^{-1}(\cos\theta + i\sin\theta) = \sin^{-1}\sqrt{\sin\theta} + i\ln\left(\sqrt{1+\sin\theta} - \sqrt{\sin\theta}\right)$

Sol Let $\alpha + i\beta = \cos^{-1}(\cos\theta + i\sin\theta)$ — (i)

(Now we find values of $\alpha$ & $\beta$ and put in (i) to get required result.)

$\therefore \cos(\alpha + i\beta) = \cos\theta + i\sin\theta$

$\cos\alpha\cos i\beta - \sin\alpha\sin i\beta = \cos\theta + i\sin\theta$

$\cos\alpha\cosh\beta - i\sin\alpha\sinh\beta = \cos\theta + i\sin\theta$

Equating Real & Imaginary parts

$\cos\alpha\cosh\beta = \cos\theta$ & $-\sin\alpha\sinh\beta = \sin\theta$

$\cosh\beta = \frac{\cos\theta}{\cos\alpha}$ — (ii) & $\sinh\beta = \frac{\sin\theta}{-\sin\alpha}$ — (iii)

Since

$\cosh^2\beta - \sinh^2\beta = 1$ (To eliminate $\beta$)

$\left(\frac{\cos\theta}{\cos\alpha}\right)^2 - \left(\frac{-\sin\theta}{\sin\alpha}\right)^2 = 1$

$\frac{\cos^2\theta\sin^2\alpha - \sin^2\theta\cos^2\alpha}{\cos^2\alpha\sin^2\alpha} = 1$

$\cos^2\theta\sin^2\alpha - \sin^2\theta\cos^2\alpha = \cos^2\alpha\sin^2\alpha$

$(1-\sin^2\theta)\sin^2\alpha - \sin^2\theta(1-\sin^2\alpha) = (1-\sin^2\alpha)\sin^2\alpha$

$\sin^2\alpha - \cancel{\sin^2\theta\sin^2\alpha} - \sin^2\theta + \cancel{\sin^2\theta\sin^2\alpha} = \sin^2\alpha - \sin^4\alpha$

$\cancel{-}\sin^2\theta = \cancel{\sin^2\alpha} - \cancel{\sin^2\alpha} \cancel{-} \sin^4\alpha$

$\sin\theta = \sin^2\alpha$

$\sqrt{\sin\theta} = \sin\alpha$

$\boxed{\sin^{-1}\sqrt{\sin\theta} = \alpha}$

Now Since $\cosh^2\beta - \sinh^2\beta = 1$

$\cosh^2\beta = 1 + \sinh^2\beta$

$\cosh\beta = \sqrt{1+\sinh^2\beta}$ — (iv)

from (iii) $\sinh\beta = \frac{\sin\theta}{-\sin\alpha} = \frac{\sin\theta}{-\sqrt{\sin\theta}} = -\sqrt{\sin\theta}$

$\therefore$ (iv) becomes $\cosh\beta = \sqrt{1+(-\sqrt{\sin\theta})^2} = \sqrt{1+\sin\theta}$

$\cosh\beta + \sinh\beta = \sqrt{1+\sin\theta} + (-\sqrt{\sin\theta})$

$\frac{e^{\beta}+e^{-\beta}}{2} + \frac{e^{\beta}-e^{-\beta}}{2} = \sqrt{1+\sin\theta} - \sqrt{\sin\theta}$

$\frac{2e^{\beta}}{2} = \sqrt{1+\sin\theta} - \sqrt{\sin\theta}$

$e^{\beta} = \sqrt{1+\sin\theta} - \sqrt{\sin\theta}$

$\beta = \ln\left[\sqrt{1+\sin\theta} - \sqrt{\sin\theta}\right]$

Put values of $\alpha$ & $\beta$ in (i)

$\cos^{-1}(\cos\theta + i\sin\theta) = \sin^{-1}\sqrt{\sin\theta} + \ln\left[\sqrt{1+\sin\theta} - \sqrt{\sin\theta}\right]$

proved

$\boxed{1\cdot 4-11}$

Q(i) Prove that $\tan^{-1}(\cos\theta + i\sin\theta) = \pm\frac{\pi}{4} + \frac{i}{4}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right)$

Sol Let $\alpha + i\beta = \tan^{-1}(\cos\theta + i\sin\theta)$ — (i)

$\alpha - i\beta = \tan^{-1}(\cos\theta - i\sin\theta)$ — (ii)

Adding (i)+(ii)

$\tan^{-1}(\cos\theta + i\sin\theta) + \tan^{-1}(\cos\theta - i\sin\theta) = 2\alpha$

$\Rightarrow \tan^{-1}\left[\frac{\cos\theta + \cancel{i\sin\theta} + \cos\theta - \cancel{i\sin\theta}}{1-(\cos\theta+i\sin\theta)(\cos\theta - i\sin\theta)}\right] = 2\alpha$

$\Rightarrow \tan^{-1}\left[\frac{2\cos\theta}{1-(\cos^2\theta+\sin^2\theta)}\right] = 2\alpha$

$\Rightarrow \tan^{-1}\left[\frac{2\cos\theta}{1-1}\right] = 2\alpha$

$\Rightarrow \tan^{-1}(\pm\infty) = 2\alpha$ as $\cos\theta > 0$ or $\cos\theta < 0$

$\Rightarrow \pm\frac{\pi}{2} = 2\alpha$

$\Rightarrow \boxed{\pm\frac{\pi}{4} = \alpha}$



Again Subtracting

$\tan^{-1}(\cos\theta + i\sin\theta) - \tan^{-1}(\cos\theta - i\sin\theta) = 2i\beta$

$\Rightarrow \tan^{-1}\left[\frac{\cancel{\cos\theta} + i\sin\theta - \cancel{\cos\theta} + i\sin\theta}{1+(\cos\theta+i\sin\theta)(\cos\theta - i\sin\theta)}\right] = 2i\beta$

$\Rightarrow \tan^{-1}\left[\frac{2i\sin\theta}{1+\cos^2\theta+\sin^2\theta}\right] = 2i\beta$

$\Rightarrow \tan^{-1}\left(\frac{\cancel{2}i\sin\theta}{\cancel{1+1}}\right) = 2i\beta$



$\Rightarrow i\sin\theta = \tan 2i\beta$

$\Rightarrow \cancel{i}\sin\theta = \cancel{i}\tanh 2\beta$

$\sin\theta = \frac{e^{2\beta} - e^{-2\beta}}{e^{2\beta} + e^{-2\beta}}$

Comp & Dividendo

$\frac{\sin\theta + 1}{\sin\theta - 1} = \frac{e^{2\beta} - \cancel{e^{-2\beta}} + e^{2\beta} + \cancel{e^{-2\beta}}}{\cancel{e^{2\beta}} - e^{-2\beta} - \cancel{e^{2\beta}} - e^{-2\beta}}$

$\frac{\sin\theta + 1}{\sin\theta - 1} = \frac{\cancel{2}e^{2\beta}}{-\cancel{2}e^{-2\beta}} = -e^{4\beta}$

$e^{4\beta} = \frac{1+\sin\theta}{1-\sin\theta}$

$4\beta = \ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right)$

$\beta = \frac{1}{4}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right)$

Put values of $\alpha$ & $\beta$ in (i) we get

$\tan^{-1}(\cos\theta + i\sin\theta) = \pm\frac{\pi}{4} + \frac{i}{4}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right)$

proved.

# Summation of Series

## Some Important Formulae:

1) $a + ar + ar^2 + \cdots\cdots ar^{n-1} = \dfrac{a(1-r^n)}{1-r}$ G. Series

2) $a + ar + ar^2 + \cdots\cdots \infty = \dfrac{a}{1-r}$ $|r|<1$ & $n \to \infty$ Infinite G. Series

3) $1 - \frac{1}{2}x + \frac{1.3}{2.4}x^2 - \frac{1.3.5}{2.4.6}x^3 \cdots\cdots \infty = \dfrac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}}$ (B. Series, $n = -\frac{1}{2}$)

4) $1 + \frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \cdots\cdots \infty = \dfrac{1}{\sqrt{1-x}} = (1-x)^{-\frac{1}{2}}$ (B. series, $n = -\frac{1}{2}$)

5) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdots \infty = \sin x$

6) $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \cdots\cdots \infty = \cos x$

7) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots \infty = e^x$

8) $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \cdots\cdots \infty = \ln(1+x)$

9) $-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} \cdots\cdots \infty = \ln(1-x)$

10) $1 + \frac{x}{2} - \frac{1}{2.4}x^2 + \frac{1.3}{2.4.6}x^3 - \frac{1.3.5}{2.4.6.8}x^4 \cdots\cdots \infty = \sqrt{1+x}$ (B Series, $n = +\frac{1}{2}$)

11) $1 - \frac{x}{2} - \frac{x^2}{2.4} - \frac{1.3}{2.4.6}x^3 - \frac{1.3.5}{2.4.6.8}x^4 \cdots\cdots \infty = \sqrt{1-x}$ (B. Series, $n = +\frac{1}{2}$)

**To Prove** $1 - e^{iA} = e^{\frac{iA}{2}}\left(-2i \sin\frac{A}{2}\right)$

**LHS** $1 - e^{iA} = e^{\frac{iA}{2}}\left(e^{-\frac{iA}{2}} - e^{\frac{iA}{2}}\right) = (2i)e^{\frac{iA}{2}}\left(\dfrac{e^{-\frac{iA}{2}} - e^{\frac{iA}{2}}}{2i}\right)$

$\dfrac{e^{\frac{iA}{2}} - e^{-\frac{iA}{2}}}{2i} = \sin\frac{A}{2}$

$\times e$, $\div$ by $2i$
Take $-$ common
$= -2i\, e^{\frac{iA}{2}}\left(\dfrac{e^{\frac{iA}{2}} - e^{-\frac{iA}{2}}}{2i}\right)$

$= -2i\, e^{\frac{iA}{2}} \sin\frac{A}{2}$.

**To Prove** $e^{\theta} = \sinh\theta + \cosh\theta$

**RHS** $= \dfrac{e^{\theta} - e^{-\theta}}{2} + \dfrac{e^{\theta} + e^{-\theta}}{2} = \dfrac{e^{\theta} - \cancel{e^{-\theta}} + e^{\theta} + \cancel{e^{-\theta}}}{2} = \dfrac{2e^{\theta}}{2} = e^{\theta} = \text{LHS}.$

# Ex 1.5



For each of problem 1–5, evaluate the indicated sum.

(i) $\sin A + \sin 2A + \sin 3A + \ldots \sin nA.$

(ii) $\cos A + \cos 2A + \cos 3A + \ldots \cos nA.$

Sol: Let $S = \sin A + \sin 2A + \sin 3A + \ldots + \sin nA.$

& $C = \cos A + \cos 2A + \cos 3A + \ldots + \cos nA$

$$\Rightarrow C + iS = (\cos A + i\sin A) + (\cos 2A + i\sin 2A) + \ldots + (\cos nA + i\sin nA)$$

$$= e^{iA} + e^{2iA} + e^{3iA} + \ldots e^{niA}$$

Geometric Series with $a = e^{iA}$, $n = n$, $r = e^{iA}$

$S_n = a\dfrac{(r^n - 1)}{r - 1}$

$$\Rightarrow C + iS = e^{iA}\frac{(e^{niA} - 1)}{e^{iA} - 1}$$

Taking $e^{\frac{niA}{2}}$ common from N

Taking $e^{\frac{iA}{2}}$ common from D

$$= e^{iA}\left\{\frac{e^{\frac{niA}{2}}\left(e^{\frac{niA}{2}} - e^{\frac{-niA}{2}}\right)}{e^{\frac{iA}{2}}\left(e^{\frac{iA}{2}} - e^{\frac{-iA}{2}}\right)}\right\}$$

$$= e^{iA} \cdot e^{\frac{niA}{2}} \cdot e^{\frac{-iA}{2}} \frac{\left(e^{\frac{niA}{2}} - e^{\frac{-niA}{2}}\right)}{\left(e^{\frac{iA}{2}} - e^{\frac{-iA}{2}}\right)}$$

$$= e^{\frac{2iA + niA - iA}{2}} \frac{\frac{\left(e^{\frac{niA}{2}} - e^{\frac{-niA}{2}}\right)}{2i}}{\frac{\left(e^{\frac{iA}{2}} - e^{\frac{-iA}{2}}\right)}{2i}}$$

÷ N & D by $2i$

$$= e^{iA\left(\frac{1+n}{2}\right)} \cdot \frac{\sin\frac{nA}{2}}{\sin\frac{A}{2}}$$

$$= \left\{\cos\left(\frac{1+n}{2}\right)A + i\sin\left(\frac{1+n}{2}\right)A\right\}\frac{\sin\frac{nA}{2}}{\sin\frac{A}{2}}$$

Comparing Real & Imaginary Parts

$$C = \cos\left(\frac{1+n}{2}\right)A \cdot \frac{\sin\frac{nA}{2}}{\sin\frac{A}{2}} \quad \& \quad S = \sin\left(\frac{1+n}{2}\right)A \cdot \frac{\sin\frac{nA}{2}}{\sin\frac{A}{2}}$$

$$\cos A + \cos 2A + \cos 3A \ldots \cos nA = \cos\left(\frac{1+n}{2}\right)A \cdot \frac{\sin\frac{nA}{2}}{\sin\frac{A}{2}}$$

&

$$\sin A + \sin 2A + \ldots \sin nA = \sin\left(\frac{n+1}{2}\right)A \cdot \frac{\sin\frac{nA}{2}}{\sin\frac{A}{2}}$$

② Let $C = \cos\theta + \cos 3\theta + \cos 5\theta + \cdots \cos(2n-1)\theta$

Sol & $S = \sin\theta + \sin 3\theta + \sin 5\theta + \cdots \sin(2n-1)\theta$

$\Rightarrow C+iS = (\cos\theta + i\sin\theta) + (\cos 3\theta + i\sin 3\theta) + \cdots\cdots (\cos(2n-1)\theta + i\sin(2n-1)\theta)$

$= e^{i\theta} + e^{3i\theta} + e^{5i\theta} + \cdots e^{(2n-1)i\theta}$

Geometric Series

$a = e^{i\theta},\ n = n,\ r = e^{2i\theta}$

$S_n = \dfrac{a(r^n - 1)}{r-1}$

$\Rightarrow C+iS = \dfrac{e^{i\theta}\left(e^{2ni\theta} - 1\right)}{\left(e^{2i\theta} - 1\right)}$

$= \cancel{e^{i\theta}}\left[\dfrac{e^{ni\theta}\left(e^{ni\theta} - e^{-ni\theta}\right)}{\cancel{e^{i\theta}}\left(e^{i\theta} - e^{-i\theta}\right)}\right]$

$= \dfrac{e^{ni\theta}\left(\dfrac{e^{ni\theta} - e^{-ni\theta}}{2i}\right)}{\left(\dfrac{e^{i\theta} - e^{-i\theta}}{2i}\right)}$ ÷ N & D by $2i$

$\Rightarrow C+iS = (\cos n\theta + i\sin n\theta)\left(\dfrac{\sin n\theta}{\sin\theta}\right)$

Comparing Real part only

$C = \cos n\theta \cdot \dfrac{\sin n\theta}{\sin\theta}$

$\cos\theta + \cos 3\theta + \cos 5\theta + \cdots \cos(2n-1)\theta = \cos n\theta \cdot \dfrac{\sin n\theta}{\sin\theta}$

$= \dfrac{2}{2}\cos n\theta \dfrac{\sin n\theta}{\sin\theta}$

$= \dfrac{\sin 2n\theta}{2\sin\theta}$

Q: $1+x\cos\theta+x^2\cos 2\theta+\cdots x^n\cos n\theta$

Sol: Let $C=1+x\cos\theta+x^2\cos 2\theta+\cdots x^n\cos n\theta$

& $S=\quad x\sin\theta+x^2\sin 2\theta+\cdots x^n\sin n\theta$

$C+iS=1+x(\cos\theta+i\sin\theta)+x^2(\cos 2\theta+i\sin 2\theta)+\cdots x^n(\cos n\theta+i\sin n\theta)$

$=1+xe^{i\theta}+x^2e^{i2\theta}+x^3e^{i3\theta}+\cdots+x^ne^{in\theta}$

G. Series
$a=1,\ r=xe^{i\theta}$
$n=n+1$
$S_n=\dfrac{a(r^n-1)}{r-1}$

$C+iS=1\left(\dfrac{(xe^{i\theta})^{n+1}-1}{xe^{i\theta}-1}\right)=\dfrac{x^{n+1}e^{(n+1)i\theta}-1}{xe^{i\theta}-1}$

$=\dfrac{x^{n+1}\{\cos(n+1)\theta+i\sin(n+1)\theta\}-1}{x(\cos\theta+i\sin\theta)-1}$

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$=\dfrac{(x^{n+1}\cos(n+1)\theta-1)+i(x^{n+1}\sin(n+1)\theta)}{(x\cos\theta-1)+i(x\sin\theta)}$

$=\dfrac{(x^{n+1}\cos(n+1)\theta-1)+i(x^{n+1}\sin(n+1)\theta)}{(x\cos\theta-1)+i(x\sin\theta)}\cdot\dfrac{(x\cos\theta-1)-i(x\sin\theta)}{(x\cos\theta-1)-i(x\sin\theta)}$

$C+iS=\dfrac{\left[(x^{n+1}\cos(n+1)\theta-1)(x\cos\theta-1)+(x^{n+1}\sin(n+1)\theta)(x\sin\theta)\right]+i\left[x^{n+1}\sin(n+1)\theta(x\cos\theta-1)-(x^{n+1}\cos(n+1)\theta-1)(x\sin\theta)\right]}{(x\cos\theta-1)^2+(x\sin\theta)^2}$

comparing Real part only

$C=\dfrac{x^{n+2}\cos(n+1)\theta\cos\theta-x^{n+1}\cos(n+1)\theta-x\cos\theta+1+x^{n+2}\sin(n+1)\theta\sin\theta}{x^2\cos^2\theta+1-2x\cos\theta+x^2\sin^2\theta}$

$=\dfrac{x^{n+2}[\cos(n+1)\theta\cos\theta+\sin(n+1)\theta\sin\theta]-x^{n+1}\cos(n+1)\theta-x\cos\theta+1}{x^2(\cos^2\theta+\sin^2\theta)-2x\cos\theta+1}$

$C=\dfrac{x^{n+2}\cos[(n+1)\theta-\theta]-x^{n+1}\cos(n+1)\theta-x\cos\theta+1}{x^2-2x\cos\theta+1}$

$C=\dfrac{x^{n+2}\cos n\theta-x^{n+1}\cos(n+1)\theta-x\cos\theta+1}{x^2-2x\cos\theta+1}$ Ans

(4) $S = 3\sin\alpha + 5\sin 2\alpha + 7\sin 3\alpha + \cdots (2n+1)\sin n\alpha$

Sol Let $S = 3\sin\alpha + 5\sin 2\alpha + 7\sin 3\alpha + \cdots (2n+1)\sin n\alpha$

& $C = 3\cos\alpha + 5\cos 2\alpha + 7\sin 3\alpha + \cdots (2n+1)\cos n\alpha$

$C+iS = 3(\cos\alpha + i\sin\alpha) + 5(\cos 2\alpha + i\sin 2\alpha) + \cdots (2n+1)(\cos n\alpha + i\sin n\alpha)$

$C+iS = 3e^{i\alpha} + 5e^{i2\alpha} + 7e^{i3\alpha} + \cdots (2n+1)e^{in\alpha}$ —— ① (Not G. Series)

Multiply eq ① by $e^{i\alpha}$ & subtract

$(C+iS)e^{i\alpha} = \quad +3e^{i2\alpha} + 5e^{i3\alpha} + 7e^{i4\alpha} \cdots (2n-1)e^{in\alpha} + (2n+1)e^{i(n+1)\alpha}$ —— ②

$(C+iS)(1-e^{i\alpha}) = 3e^{i\alpha} + 2e^{i2\alpha} + 2e^{i3\alpha} + \cdots + 2e^{in\alpha} - (2n+1)e^{i(n+1)\alpha}$

$= (e^{i\alpha} + 2e^{i\alpha}) + 2e^{i2\alpha} + 2e^{i3\alpha} + \cdots 2e^{in\alpha} - (2n+1)e^{i(n+1)\alpha}$

$= e^{i\alpha} + 2(e^{i\alpha} + e^{i2\alpha} + e^{i3\alpha} + \cdots e^{in\alpha}) - (2n+1)e^{i(n+1)\alpha}$

$= e^{i\alpha} + 2\left[e^{i\alpha}\dfrac{(e^{ni\alpha} - 1)}{(e^{i\alpha} - 1)}\right] - (2n+1)e^{i(n+1)\alpha}$    G. Series $a = e^{i\alpha}$, $r = e^{i\alpha}$, $n = n$

LCM $= \dfrac{e^{i\alpha}(e^{i\alpha}-1) + 2e^{i\alpha}(e^{ni\alpha} - 1) - (2n+1)e^{i(n+1)\alpha}\cdot(e^{i\alpha}-1)}{(e^{i\alpha}-1)}$

$(C+iS)(1-e^{i\alpha})(e^{i\alpha}-1) = e^{i\alpha}\left[e^{i\alpha} - 1 + 2e^{ni\alpha} - 2 - (2n+1)e^{i(n+1)\alpha} + (2n+1)e^{ni\alpha}\right]$ ✓

$-(C+iS)(e^{i\alpha}-1)^2 = e^{i\alpha}\left[e^{i\alpha} - 3 + (2+2n+1)e^{ni\alpha} - (2n+1)e^{i(n+1)\alpha}\right]$

$-(C+iS)\left[-4e^{i\alpha}\sin^2\frac{\alpha}{2}\right] = e^{i\alpha}\left[e^{i\alpha} - 3 + (2n+3)e^{ni\alpha} - (2n+1)e^{i(n+1)\alpha}\right]$

$(C+iS) = \dfrac{\cancel{e^{i\alpha}}\left(e^{i\alpha} - 3 + (2n+3)e^{ni\alpha} - (2n+1)e^{i(n+1)\alpha}\right)}{4\cancel{e^{i\alpha}}\sin^2\frac{\alpha}{2}}$

$(e^{i\alpha} - 1)^2$

$= \left[e^{\frac{i\alpha}{2}}\left(e^{\frac{i\alpha}{2}} - e^{-\frac{i\alpha}{2}}\right)\right]^2$

$= \left[e^{\frac{i\alpha}{2}}\left(\dfrac{e^{\frac{i\alpha}{2}} - e^{-\frac{i\alpha}{2}}}{2i}\right)2i\right]^2$

$= e^{\left(\frac{i\alpha}{2}\right)^2}\sin^2\frac{\alpha}{2}(-4)$

$= -4e^{i\alpha}\sin^2\frac{\alpha}{2}$

$= \dfrac{\cos\alpha + i\sin\alpha - 3 + (2n+3)(\cos n\alpha + i\sin n\alpha) - (2n+1)(\cos(n+1)\alpha + i\sin(n+1)\alpha)}{4\left(\frac{1-\cos\alpha}{2}\right)}$

Comparing Imaginary Part

$S = \dfrac{\sin\alpha + (2n+3)\sin n\alpha - (2n+1)\sin(n+1)\alpha}{2(1-\cos\alpha)}$

(5) $\cos^2\theta + \cos^2 2\theta + \cos^2 3\theta + \cdots \cos^2 n\theta$

$$= \frac{1+\cos 2\theta}{2} + \frac{1+\cos 4\theta}{2} + \frac{1+\cos 6\theta}{2} + \cdots \frac{1+\cos 2n\theta}{2}$$

$$= \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots n \text{ terms}\right) + \frac{\cos 2\theta}{2} + \frac{\cos 4\theta}{2} + \frac{\cos 6\theta}{2} \cdots \frac{\cos 2n\theta}{2}$$

$$S_n = \frac{n}{2} + \frac{1}{2}(\cos 2\theta + \cos 4\theta + \cos 6\theta + \cdots \cos 2n\theta) \quad \text{—(1)}$$

Let $C = \cos 2\theta + \cos 4\theta + \cos 6\theta + \cdots \cos 2n\theta$

$S = \sin 2\theta + \sin 4\theta + \sin 6\theta + \cdots \sin 2n\theta$

$$C + iS = e^{2i\theta} + e^{4i\theta} + e^{6i\theta} + \cdots e^{2ni\theta}$$

Geometric Series
$a = e^{2i\theta}$, $r = e^{2i\theta}$, $n = n$

$S_n = \dfrac{a(r^n - 1)}{r - 1}$

$$C + iS = \frac{e^{2i\theta}\left(e^{i2\theta n} - 1\right)}{e^{2i\theta} - 1}$$

$$= \frac{e^{2i\theta} \cdot e^{ni\theta}\left(e^{ni\theta} - e^{-ni\theta}\right)}{e^{i\theta}\left(e^{i\theta} - e^{-i\theta}\right)}$$

$$= e^{i\theta} \cdot e^{ni\theta} \frac{\left(e^{ni\theta} - e^{-ni\theta}\right)}{\left(e^{i\theta} - e^{-i\theta}\right)}$$

$$= e^{(n+1)i\theta} \frac{\left(\frac{e^{ni\theta} - e^{-ni\theta}}{2i}\right)}{\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)}$$

$$= e^{(n+1)i\theta} \cdot \frac{\sin n\theta}{\sin\theta}$$

$$C + iS = (\cos(n+1)\theta + i\sin(n+1)\theta)\frac{\sin n\theta}{\sin\theta}$$

Comparing Real Part

$$C = \cos(n+1)\theta \, \frac{\sin n\theta}{\sin\theta}$$

Hence from (1) $S_n = \dfrac{n}{2} + \dfrac{1}{2}\dfrac{\sin n\theta}{\sin\theta}\cos(n+1)\theta$

⑥ Find sum of Infinite series.

Let $S = \sin\theta + \frac{1}{2}\sin 3\theta + \frac{1\cdot 3}{2\cdot 4}\sin 5\theta + \cdots\cdots \infty$

$C = \cos\theta + \frac{1}{2}\cos 3\theta + \frac{1\cdot 3}{2\cdot 4}\cos 5\theta + \cdots\cdots \infty$

$$C+iS = e^{i\theta} + \frac{1}{2}e^{3i\theta} + \frac{1\cdot 3}{2\cdot 4}e^{5i\theta} + \cdots\cdots \infty$$

$$= e^{i\theta}\left(1+\frac{1}{2}e^{2i\theta} + \frac{1\cdot 3}{2\cdot 4}e^{4i\theta} + \cdots\cdots \infty\right)$$

$$= e^{i\theta}\left(1-e^{2i\theta}\right)^{-\frac{1}{2}} \qquad \because (1-x)^{-\frac{1}{2}} = \frac{1}{\sqrt{(1-x)}} = 1+\frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \cdots$$

$$= e^{i\theta}\left(e^{2i\theta}\right)^{-\frac{1}{2}}\left(e^{-2i\theta}-1\right)^{-\frac{1}{2}}$$

$$= e^{i\theta}e^{-i\theta}\left(\cos(-2\theta)+i\sin(-2\theta)-1\right)^{-\frac{1}{2}}$$

$$= e^{0}\left(\cos 2\theta - i\sin 2\theta - 1\right)^{-\frac{1}{2}}$$

$$= \left[(\cos 2\theta - 1) - i\sin 2\theta\right]^{-\frac{1}{2}} \qquad \cos 2\theta = 1-2\sin^2\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= \left(-2\sin^2\theta - i2\sin\theta\cos\theta\right)^{-\frac{1}{2}}$$

$$= \left\{2\sin\theta(-\sin\theta - i\cos\theta)\right\}^{-\frac{1}{2}}$$

$$= (2\sin\theta)^{-\frac{1}{2}}\left(-\sin\theta - i\cos\theta\right)^{-\frac{1}{2}}$$

$$= (2\sin\theta)^{-\frac{1}{2}}\left[\cos\left(\frac{\pi}{2}+\theta\right) - i\sin\left(\frac{\pi}{2}+\theta\right)\right]^{-\frac{1}{2}} \qquad \cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta, \quad \sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$$

$$= (2\sin\theta)^{-\frac{1}{2}}\left[\cos\left(-\frac{\pi}{4}-\frac{\theta}{2}\right) - i\sin\left(-\frac{\pi}{4}-\frac{\theta}{2}\right)\right]$$

$$C+iS = (2\sin\theta)^{-\frac{1}{2}}\left[\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right) + i\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$$

Comparing Imaginary parts

$$S = (2\sin\theta)^{-\frac{1}{2}}\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

$$S = \frac{\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}{\sqrt{2\sin\theta}} \quad \text{Ans.}$$

(7) $\sinh\theta + \frac{\sinh 2\theta}{\lfloor 2} + \frac{\sinh 3\theta}{\lfloor 3} + \cdots\cdots\infty$

Sol $= \left(\frac{e^{\theta}-e^{-\theta}}{2}\right) + \left(\frac{e^{2\theta}-e^{-2\theta}}{2\lfloor 2}\right) + \left(\frac{e^{3\theta}-e^{-3\theta}}{2\lfloor 3}\right) + \cdots\cdots\infty$

$= \frac{1}{2}\left[e^{\theta} - e^{-\theta} + \frac{e^{2\theta}-e^{-2\theta}}{\lfloor 2} + \frac{e^{3\theta}-e^{-3\theta}}{\lfloor 3} + \cdots\cdots\infty\right]$

$= \frac{1}{2}\left[e^{\theta} + \frac{e^{2\theta}}{\lfloor 2} + \frac{e^{3\theta}}{\lfloor 3} + \cdots\infty\right] - \frac{1}{2}\left[e^{-\theta} + \frac{e^{-2\theta}}{\lfloor 2} + \frac{e^{-3\theta}}{\lfloor 3} + \cdots\infty\right]$

Add & Subtract 1

$\Rightarrow \frac{1}{2}\left[1 + e^{\theta} + \frac{e^{2\theta}}{\lfloor 2} + \frac{e^{3\theta}}{\lfloor 3} + \cdots\infty\right] - \frac{1}{2}\left[1 + e^{-\theta} + \frac{e^{-2\theta}}{\lfloor 2} + \frac{e^{-3\theta}}{\lfloor 3} + \cdots\infty\right]$

$\because e^{x} = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots\infty$

$= \frac{1}{2}\left(e^{e^{\theta}}\right) - \frac{1}{2}\left(e^{e^{-\theta}}\right)$

$= \frac{1}{2}\left[e^{e^{\theta}} - e^{e^{-\theta}}\right]$

* $= \frac{1}{2}\left[e^{\cosh\theta + \sinh\theta} - e^{\cosh\theta - \sinh\theta}\right]$

$= \frac{e^{\cosh\theta}}{2}\left(e^{\sinh\theta} - e^{-\sinh\theta}\right)$

$= e^{\cosh\theta}\left(\frac{e^{\sinh\theta} - e^{-\sinh\theta}}{2}\right)$

$= e^{\cosh\theta}\sinh(\sinh\theta)$ Ans

* $\cosh\theta + \sinh\theta = e^{\theta}$

LHS $= \frac{e^{\theta}+e^{-\theta}}{2} + \frac{e^{\theta}-e^{-\theta}}{2} = \frac{e^{\theta} + \cancel{e^{-\theta}} + e^{\theta} - \cancel{e^{-\theta}}}{2}$

$= \frac{2e^{\theta}}{2} = e^{\theta}$ RHS

$=$

Q13 $C = 1 + c\cos\theta + \frac{c^2}{\lfloor 2}\cos 2\theta + \frac{c^3}{\lfloor 3}\cos 3\theta + \cdots\cdots$

$S = c\sin\theta + \frac{c^2}{\lfloor 2}\sin 2\theta + \frac{c^3}{\lfloor 3}\sin 3\theta + \cdots\cdots$

$C + iS = 1 + c(\cos\theta + i\sin\theta) + \frac{c^2}{\lfloor 2}(\cos 2\theta + i\sin 2\theta) + \cdots\cdots$

$= 1 + ce^{i\theta} + \frac{c^2}{\lfloor 2}e^{i2\theta} + \frac{c^3}{\lfloor 3}e^{3i\theta} + \cdots\cdots$

$= 1 + ce^{i\theta} + \frac{(ce^{i\theta})^2}{\lfloor 2} + \frac{(ce^{i\theta})^3}{\lfloor 3} + \cdots\cdots$

$= e^{ce^{i\theta}}$ $\because e^{x} = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots\infty$

$= e^{c(\cos\theta + i\sin\theta)}$

$= e^{c\cos\theta} \cdot e^{ic\sin\theta}$

$C + iS = e^{c\cos\theta}\left(\cos(c\sin\theta) + i\sin(c\sin\theta)\right)$

Comparing Real Part

$C = e^{c\cos\theta}\cos(c\sin\theta)$ Ans.

Q8 $S = \sin\alpha \cdot \sin\alpha + \sin^2\alpha \cdot \sin 2\alpha + \sin^3\alpha \cdot \sin 3\alpha + \cdots\cdots \infty$

$C = \sin\alpha \cdot \cos\alpha + \sin^2\alpha \cdot \cos 2\alpha + \sin^3\alpha \cdot \cos 3\alpha + \cdots\cdots \infty$

$C + iS = \sin\alpha(\cos\alpha + i\sin\alpha) + \sin^2\alpha(\cos 2\alpha + i\sin 2\alpha) + \cdots\cdots \infty$

$= \sin\alpha\, e^{i\alpha} + \sin^2\alpha\, e^{2i\alpha} + \sin^3\alpha\, e^{3i\alpha} + \cdots \infty$ Infinite Geometric Series

$a = \sin\alpha\, e^{i\alpha}$

$r = \sin\alpha\, e^{i\alpha}$

$S_\infty = \dfrac{a}{1-r}$

$$C + iS = \frac{a}{1-r} = \frac{\sin\alpha\, e^{i\alpha}}{1 - \sin\alpha\, e^{i\alpha}}$$

$$= \frac{\sin\alpha\, \cancel{e^{i\alpha}}}{\cancel{e^{i\alpha}}(e^{-i\alpha} - \sin\alpha)}$$

$$= \frac{\sin\alpha}{(\cos\alpha - i\sin\alpha) - \sin\alpha}$$

$$= \frac{\sin\alpha}{(\cos\alpha - \sin\alpha) - i\sin\alpha}$$

$$= \frac{\sin\alpha}{(\cos\alpha - \sin\alpha) - i\sin\alpha} \cdot \frac{[(\cos\alpha - \sin\alpha) + i\sin\alpha]}{[(\cos\alpha - \sin\alpha) + i\sin\alpha]}$$

$$= \frac{\sin\alpha\,[(\cos\alpha - \sin\alpha) + i\sin\alpha]}{(\cos\alpha - \sin\alpha)^2 + \sin^2\alpha}$$

$$= \frac{\sin\alpha(\cos\alpha - \sin\alpha) + i\sin^2\alpha}{\cos^2\alpha + \sin^2\alpha - 2\sin\alpha\cos\alpha + \sin^2\alpha}$$

$$C + iS = \frac{\sin\alpha(\cos\alpha - \sin\alpha) + i\sin^2\alpha}{1 - \sin 2\alpha + \sin^2\alpha}$$

Comparing imaginary parts

$$S = \frac{\sin^2\alpha}{1 - \sin 2\alpha + \sin^2\alpha}$$ Ans.

1.5-10

Q(9) $C = 1 - \frac{1}{2}\cos\theta + \frac{1.3}{2.4}\cos 2\theta - \frac{1.3.5}{2.4.6}\cos 3\theta + \cdots \infty$

$S = -\frac{1}{2}\sin\theta + \frac{1.3}{2.4}\sin 2\theta - \frac{1.3.5}{2.4.6}\sin 3\theta + \cdots \infty$

$C+iS = 1 - \frac{1}{2}(\cos\theta + i\sin\theta) + \frac{1.3}{2.4}(\cos 2\theta + i\sin 2\theta) - \cdots \infty$

$= 1 - \frac{1}{2}e^{i\theta} + \frac{1.3}{2.4}e^{2i\theta} - \frac{1.3.5}{2.4.6}e^{3i\theta} + \cdots \infty$

$= (1+e^{i\theta})^{-\frac{1}{2}}$ $\quad \because (1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{1.3}{2.4}x^2 - \frac{1.3.5}{2.4.6}x^3 + \cdots \infty$ B.Series

$= (1+\cos\theta + i\sin\theta)^{-\frac{1}{2}}$

$= \left(2\cos^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-\frac{1}{2}}$

$= \left[2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\right]^{-\frac{1}{2}}$

DeMoivres

$= \left(2\cos\frac{\theta}{2}\right)^{-\frac{1}{2}}\left(\cos\frac{\theta}{4} - i\sin\frac{\theta}{4}\right)$

$$C+iS = \frac{\cos\frac{\theta}{4} - i\sin\frac{\theta}{4}}{\sqrt{2\cos\frac{\theta}{2}}}$$

Comparing Real Part

$$C = \frac{\cos\frac{\theta}{4}}{\sqrt{2\cos\frac{\theta}{2}}} \quad \text{Ans}$$

x———————x

Q(11) $C = 1 + \frac{1}{2}\cos\theta + \frac{1.3}{2.4}\cos 2\theta + \frac{1.3.5}{2.4.6}\cos 3\theta + \cdots \infty$

$S = \frac{1}{2}\sin\theta + \frac{1.3}{2.4}\sin 2\theta + \frac{1.3.5}{2.4.6}\sin 3\theta + \cdots \infty$

$C+iS \quad 1 + \frac{1}{2}e^{i\theta} + \frac{1.3}{2.4}e^{2i\theta} + \frac{1.3.5}{2.4.6}e^{3i\theta} + \cdots \infty$

$= (1-e^{i\theta})^{-\frac{1}{2}}$ $\quad \because (1-x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 \cdots$ B.Series

$= (1 - \cos\theta - i\sin\theta)^{-\frac{1}{2}}$

$= \left(2\sin^2\frac{\theta}{2} - i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-\frac{1}{2}}$

$= \left(2\sin\frac{\theta}{2}\right)^{-\frac{1}{2}}\left(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right)^{-\frac{1}{2}}$

$= \left(2\sin\frac{\theta}{2}\right)^{-\frac{1}{2}}\left[\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) - i\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right]^{-\frac{1}{2}}$

$= \left(2\sin\frac{\theta}{2}\right)^{-\frac{1}{2}}\left(\cos\left(\frac{\pi-\theta}{4}\right) + i\sin\left(\frac{\pi-\theta}{4}\right)\right)$

DeMoivres Th.

Comparing Real Part

$C = \left(2\sin\frac{\theta}{2}\right)^{-\frac{1}{2}}\left(\cos\left(\frac{\pi-\theta}{4}\right)\right)$

$$C = \frac{\cos\left(\frac{\pi-\theta}{4}\right)}{\sqrt{2\sin\frac{\theta}{2}}} \quad \text{Ans}$$

x———————x

(10) $S = n\sin\theta + \frac{n(n+1)}{\underline{|2}}\sin 2\theta + \frac{n(n+1)(n+2)}{\underline{|3}}\sin 3\theta + \cdots\cdots\infty$

$C = 1 + n\cos\theta + \frac{n(n+1)}{\underline{|2}}\cos 2\theta + \frac{n(n+1)(n+2)}{\underline{|3}}\cos 3\theta + \cdots\cdots\infty$

$$C+iS = 1 + n(\cos\theta + i\sin\theta) + \frac{n(n+1)}{\underline{|2}}(\cos 2\theta + i\sin 2\theta) + \frac{n(n+1)(n+2)}{\underline{|3}}(\cos 3\theta + i\sin 3\theta) + \cdots$$

$$= 1 + n e^{i\theta} + \frac{n(n+1)}{\underline{|2}} e^{2i\theta} + \frac{n(n+1)(n+2)}{\underline{|3}} e^{3i\theta} + \cdots\cdots\infty$$

$$= 1 + (-n)(-e^{i\theta}) + \frac{(-n)(-n-1)}{\underline{|2}}(-e^{i\theta})^2 + \frac{(-n)(-n-1)(-n-2)}{\underline{|3}}(-e^{i\theta})^3 + \cdots\cdots\infty$$

$$= \left(1 - e^{i\theta}\right)^{-n}$$

$\because$ B. Series $(1-x)^{-n} = 1 + nx + \frac{n(n-1)}{\underline{|2}}x^2 + \cdots\cdots\infty$

$$= \left(1 - \cos\theta - i\sin\theta\right)^{-n}$$

$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$

$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$$= \left(2\sin^2\frac{\theta}{2} - i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-n}$$

$$= \left(2\sin\frac{\theta}{2}\right)^{-n}\left[\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right]^{-n}$$

$\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \sin\frac{\theta}{2}$

$\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \cos\frac{\theta}{2}$

$$= \left(2\sin\frac{\theta}{2}\right)^{-n}\left[\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) - i\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right]^{-n}$$

Demoivres Th.

$$= \left(2\sin\frac{\theta}{2}\right)^{-n}\left[\cos\left(-n\left(\frac{\pi-\theta}{2}\right)\right) - i\sin\left(-n\left(\frac{\pi-\theta}{2}\right)\right)\right]$$

$$= \left(2\sin\frac{\theta}{2}\right)^{-n}\left[\cos\left(\frac{n\pi - n\theta}{2}\right) + i\sin\left(\frac{n\pi - n\theta}{2}\right)\right]$$

Comparing Imaginary parts

$$S = \left(2\sin\frac{\theta}{2}\right)^{-n}\sin\left(\frac{n\pi - n\theta}{2}\right)$$

$$S = \frac{\sin\frac{n}{2}(\pi-\theta)}{\left(2\sin\frac{\theta}{2}\right)^n}$$ Ans.

1·5-12

(13)

$$C = \cos\alpha - \frac{\cos(\alpha+2\beta)}{|\underline{3}} + \frac{\cos(\alpha+4\beta)}{|\underline{5}} + \cdots\cdots \infty$$

$$S = \sin\alpha - \frac{\sin(\alpha+2\beta)}{|\underline{3}} + \frac{\sin(\alpha+4\beta)}{|\underline{5}} + \cdots \infty$$

$$C+iS = e^{i\alpha} - \frac{e^{i(\alpha+2\beta)}}{|\underline{3}} + \frac{e^{i(\alpha+4\beta)}}{|\underline{5}} + \cdots\cdots \infty$$

$$= e^{i\alpha}\left(1 - \frac{e^{i2\beta}}{|\underline{3}} + \frac{e^{i4\beta}}{|\underline{5}} - \cdots\cdots \infty\right)$$

$$= e^{i\alpha} \cdot \frac{e^{i\beta}}{e^{i\beta}}\left(1 - \frac{e^{i2\beta}}{|\underline{3}} + \frac{e^{i4\beta}}{|\underline{5}} - \cdots \infty\right)$$

$$= \frac{e^{i\alpha}}{e^{i\beta}}\left(e^{i\beta} - \frac{e^{i3\beta}}{|\underline{3}} + \frac{e^{i5\beta}}{|\underline{5}} - \cdots\cdots \infty\right)$$

$$= e^{i(\alpha-\beta)}\left(\sin e^{i\beta}\right) \qquad \because \sin x = x - \frac{x^3}{|\underline{3}} + \frac{x^5}{|\underline{5}} - \frac{x^7}{|\underline{7}} + \cdots\cdots \infty$$

$$= \left(\cos(\alpha-\beta) + i\sin(\alpha-\beta)\right) \sin e^{i\beta}$$

$$= \left(\cos(\alpha-\beta) + i\sin(\alpha-\beta)\right) \sin(\cos\beta + i\sin\beta)$$

$$= \left(\cos(\alpha-\beta) + i\sin(\alpha-\beta)\right)\left[\sin(\cos\beta)\cos(i\sin\beta) + \cos(\cos\beta)\sin(i\sin\beta)\right]$$

$$= \left[\cos(\alpha-\beta) + i\sin(\alpha-\beta)\right]\left[\sin(\cos\beta)\cos(i\sin\beta) + \cos(\cos\beta)\sin(i\sin\beta)\right]$$

$$= \left[\cos(\alpha-\beta) + i\sin(\alpha-\beta)\right]\left[\sin(\cos\beta)\cosh(\sin\beta) + i\cos(\cos\beta)\sinh(\sin\beta)\right]$$

$$C+iS = \left[\cos(\alpha-\beta)\sin(\cos\beta)\cosh(\sin\beta) - \sin(\alpha-\beta)\cos(\cos\beta)\sinh(\sin\beta)\right]$$
$$+ i\left[\sin(\alpha-\beta)\sin\cos\beta\cosh(\sin\beta) + \cos(\alpha-\beta)\cos(\cos\beta)\sinh(\sin\beta)\right]$$

Comparing Real Part

$$C = \cos(\alpha-\beta)\sin(\cos\beta)\cosh(\sin\beta) - \sin(\alpha-\beta)\cos(\cos\beta)\sinh(\sin\beta)$$

Ans.

1-5-13

Q14 $C = C\cos\theta + \frac{c^2}{2}\cos 2\theta + \frac{c^3}{3}\cos 3\theta + \cdots$

$S = c\sin\theta + \frac{c^2}{2}\sin 2\theta + \frac{c^3}{3}\sin 3\theta + \cdots$

$$C+iS = c(\cos\theta + i\sin\theta) + \frac{c^2}{2}(\cos 2\theta + i\sin 2\theta) + \frac{c^3}{3}(\cos 3\theta + i\sin 3\theta) + \cdots$$

$$= c\,e^{i\theta} + \frac{c^2}{2}e^{i2\theta} + \frac{c^3}{3}e^{i3\theta} + \cdots$$

$$= ce^{i\theta} + \frac{(ce^{i\theta})^2}{2} + \frac{(ce^{i\theta})^3}{3} + \cdots$$

$$= -\ln(1-ce^{i\theta}) \qquad \because \ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

$$= -\operatorname{Log}(1-ce^{i\theta}) \qquad \ln(1-x) = -\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right.$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

$$= -\log(\overline{1 - c\cos\theta - ic\sin\theta})$$

$$\because \operatorname{Log}(x+iy) = \ln|x+iy| + i\operatorname{Arg} z$$

$$= -\left[\ln\left|\overline{1-c\cos\theta} - ic\sin\theta\right| + i\arg(\overline{1-c\cos\theta} - ic\sin\theta)\right.$$

$$c+iS = -\left[\ln\sqrt{(1-c\cos\theta)^2 + c^2\sin^2\theta} + i\tan^{-1}\left(-\frac{c\sin\theta}{1-c\cos\theta}\right)\right]$$

Comparing Real Part

$$C = -\left[\ln\sqrt{(1-c\cos\theta)^2 + c^2\sin^2\theta}\right]$$

$$= -\frac{1}{2}\ln\{(1-c\cos\theta)^2 + c^2\sin^2\theta\}$$

$$= -\frac{1}{2}\ln(1 + c^2\cos^2\theta - 2c\cos\theta + c^2\sin^2\theta)$$

$$= -\frac{1}{2}\ln(1 + c^2(\cos^2\theta + \sin^2\theta) - 2c\cos\theta)$$

$$= -\frac{1}{2}\ln(1 + c^2 - 2c\cos\theta) \quad \text{Ans}$$

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(15) $S = \sin\theta - \frac{1}{2}\sin 3\theta + \frac{1}{3}\sin 5\theta + \cdots$

$C = \cos\theta - \frac{1}{2}\cos 3\theta + \frac{1}{3}\cos 5\theta + \cdots$

$C + iS = e^{i\theta} - \frac{1}{2}e^{i3\theta} + \frac{1}{3}e^{i5\theta}, \cdots$

$x + iy^{i\theta}$. $= \frac{e^{i\theta}}{e^{i\theta}}\left(e^{i\theta} - \frac{1}{2}e^{i3\theta} + \frac{1}{3}e^{i5\theta} \cdots\right)$

$= \frac{1}{e^{i\theta}}\left(e^{2i\theta} - \frac{1}{2}e^{4i\theta} + \frac{1}{3}e^{6i\theta} \cdots\right)$

$= e^{-i\theta}\left[e^{2i\theta} - \frac{1}{2}(e^{2i\theta})^2 + \frac{1}{3}(e^{2i\theta})^3 \cdots\right]$

$= e^{-i\theta}\left[\log(1 + e^{2i\theta})\right]$ $\quad \because \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \infty$

$= e^{-i\theta}\left[\log(1 + \cos 2\theta + i\sin 2\theta)\right]$

$= e^{-i\theta}\left[\ln\sqrt{(1+\cos 2\theta)^2 + (\sin 2\theta)^2} + i\tan^{-1}\frac{\sin 2\theta}{1+\cos 2\theta}\right]$

$= e^{-i\theta}\left[\ln\sqrt{1 + \cos^2 2\theta + 2\cos 2\theta + \sin^2 2\theta} + i\tan^{-1}\left(\frac{\cancel{2}\sin\theta\cancel{\cos\theta}}{\cancel{2}\cos^{\cancel{2}}\theta}\right)\right]$

$= e^{-i\theta}\left[\ln\sqrt{1 + 1 + 2\cos 2\theta} + i\tan^{-1}\tan\theta\right]$

$= e^{-i\theta}\left[\ln\sqrt{2(1+\cos 2\theta)} + i\theta\right]$

$= e^{-i\theta}\left[\ln\sqrt{2(2\cos^2\theta)} + i\theta\right]$

$= e^{-i\theta}\left[\ln(2\cos\theta) + i\theta\right]$

$= (\cos\theta - i\sin\theta)(\ln 2\cos\theta + i\theta)$

$= \cos\theta\ln 2\cos\theta + \theta\sin\theta + i(\theta\cos\theta - \sin\theta\ln(2\cos\theta))$

Comparing Imaginary parts

$S = \theta\cos\theta - \sin\theta\ln(2\cos\theta)$ Ans

## Ex 1.2



① Write the following expression in the form of $a+ib$.

(i) $(-\sqrt{3}+i)^2$

Let $z = (-\sqrt{3}+i)$ $\Rightarrow x = -\sqrt{3}$, $y = 1$

$r = |z| = \sqrt{(-\sqrt{3})^2+1^2} = \sqrt{3+1} = 2$

$\cos\theta = \dfrac{x}{r} = \dfrac{-\sqrt{3}}{2}$ $\Rightarrow \theta = \cos^{-1}\left(\dfrac{-\sqrt{3}}{2}\right)$

$\sin\theta = \dfrac{y}{r} = \dfrac{1}{2}$ $\Rightarrow \theta = \sin^{-1}\left(\dfrac{1}{2}\right)$ $\Bigg\}$ $\theta = \dfrac{5\pi}{6}$

$\because$ $\left.\begin{array}{l} x \text{ is } -ve \\ y \text{ is } +ve \end{array}\right\}$ So $\theta$ lies in 2nd Quad.

$\therefore$ Principal Arg of $z = \pi - \theta$

$= \pi - \dfrac{\pi}{6} = \dfrac{5\pi}{6}$

Hence $z = r(\cos\theta + i\sin\theta)$

$-\sqrt{3}+i = 2\left(\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}\right)$

squaring $(-\sqrt{3}+i)^2 = 2^2\left(\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}\right)^2$

$= 4\left(\cos 2\left(\dfrac{5\pi}{6}\right) + i\sin 2\left(\dfrac{5\pi}{6}\right)\right)$

$= 4\left(\cos\dfrac{5\pi}{3} + i\sin\dfrac{5\pi}{3}\right)$

$= 4\left(\cos\left(\dfrac{-\pi}{3}\right) + i\sin\left(\dfrac{-\pi}{3}\right)\right)$

$= 4\left(\cos\dfrac{\pi}{3} - i\sin\dfrac{\pi}{3}\right)$

$= 4\left(\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}\right)$

$= 2 - i2\sqrt{3}$

$\dfrac{5\pi}{3} - 2\pi = \dfrac{-\pi}{3}$

$\cos\dfrac{\pi}{3} = \dfrac{1}{2}$

$\sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$

(ii) $(-3i)^4$

Let $z = -3i$ $\Rightarrow x = 0$

$= 0 - 3i$ $\quad y = -3$

$r = |z| = \sqrt{0^2+(-3)^2} = 3.$

$\cos\theta = \dfrac{x}{r} = \dfrac{0}{3} = 0$ $\Rightarrow \theta = \cos^{-1}(0)$

$\sin\theta = \dfrac{y}{r} = \dfrac{-3}{3} = -1$ $\Rightarrow \theta = \cos^{-1}(-1)$ $\Bigg\}$ $\Rightarrow \theta = \dfrac{-\pi}{2}$

$\because$ $\left.\begin{array}{l} x \text{ is } +ve \\ y \text{ is } -ve \end{array}\right\}$ So $\theta$ lies in $4^{th}$ Quad.

$\therefore$ Principal Arg of $z = -\theta$

$= -\dfrac{\pi}{2}$

$$z = r(\cos\theta + i\sin\theta)$$

$$-3i = 3\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$$

$$(-3i)^4 = 3^4\left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]^4$$

$$= 81\left[\cos\left(4\cdot\frac{\pi}{2}\right) + i\sin\left(-4\cdot\frac{\pi}{2}\right)\right]$$

$$= 81(\cos(2\pi) + i\sin(-2\pi))$$

$$= 81(\cos 2\pi - i\sin 2\pi)$$

$$= 81(1-0)$$

$$(-3i)^4 = 81 \text{ Ans}$$



(iii) $\left(\frac{1-\sqrt{3}i}{1+\sqrt{3}i}\right)^6$

Let $z = \frac{1-\sqrt{3}i}{1+\sqrt{3}i}$

$$= \frac{1-\sqrt{3}i}{1+\sqrt{3}i} \times \frac{1-\sqrt{3}i}{1-\sqrt{3}i}$$

$$= \frac{(1-\sqrt{3}i)^2}{1+3}$$

$$= \frac{1^2 + (\sqrt{3}i)^2 - 2\cdot1\cdot\sqrt{3}i}{4}$$

$$= \frac{1-3-2\sqrt{3}i}{4}$$

$$= -\frac{2}{4} - \frac{2\sqrt{3}}{4}i$$

$$z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$\left.\begin{array}{l} x = -\frac{1}{2} \\ y = -\frac{\sqrt{3}}{2}\end{array}\right\} \Rightarrow r = |z| = \sqrt{\frac{1}{4} + \frac{3}{4}}$ $\quad \boxed{r = 1}$

$\left.\begin{array}{ll} \cos\theta = \frac{x}{r} = \frac{-1/2}{1} & \Rightarrow \theta = \cos^{-1}\left(-\frac{1}{2}\right) \\ \sin\theta = \frac{y}{r} = -\frac{\sqrt{3}}{2} & \Rightarrow \theta = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\end{array}\right\} \Rightarrow \theta = -\frac{2\pi}{3}$

$\because$ $x$ is -ive, $y$ is -ive } So $\theta$ lies in III$^{rd}$ Quad
So Principal Arg of $z = -(\pi - \theta)$
$= -\left(\pi - \frac{\pi}{3}\right) = \boxed{-\frac{2\pi}{3}}$

$$\therefore z = r(\cos\theta + i\sin\theta)$$

$$\therefore \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 1\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$$

So $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^6 = 1^6\left[\cos 6\left(-\frac{2\pi}{3}\right)+i\sin 6\left(-\frac{2\pi}{3}\right)\right]$

$= \cos(-4\pi)+i\sin(-4\pi)$ $\quad \cos(-\theta)=\cos\theta$, $\sin(-\theta)=-\sin\theta$

$= \cos 4\pi - i\sin 4\pi$

$= 1-0 = 1$ ans.

**Q. No. 2**

**Part-(i)** Simplify $\dfrac{(\cos 2\theta+i\sin 2\theta)^5(\cos 3\theta-i\sin 3\theta)^6}{(\cos 4\theta-i\sin 4\theta)^7(\cos 5\theta+i\sin 5\theta)^8}$

**Sol**

$$= \frac{(\cos 2\theta+i\sin 2\theta)^5(\cos(-3\theta)+i\sin(-3\theta))^6}{[\cos(-4\theta)+i\sin(-4\theta)]^7[\cos 5\theta+i\sin 5\theta]^8}$$

$$= \frac{[(\cos\theta+i\sin\theta)^2]^5[(\cos\theta+i\sin\theta)^{-3}]^6}{[(\cos\theta+i\sin\theta)^{-4}]^7[(\cos\theta+i\sin\theta)^5]^8}$$

$$= \frac{(\cos\theta+i\sin\theta)^{10}(\cos\theta+i\sin\theta)^{-18}}{(\cos\theta+i\sin\theta)^{-28}(\cos\theta+i\sin\theta)^{40}}$$

$$= (\cos\theta+i\sin\theta)^{10-18+28-40}$$

$$= (\cos\theta+i\sin\theta)^{-20}$$

$$= \cos(-20\theta)+i\sin(-20\theta)$$

$$= \cos 20\theta - i\sin 20\theta \quad \text{ans.}$$

**Part-(ii)** $\dfrac{(\cos\alpha-i\sin\alpha)^{11}}{(\cos\beta+i\sin\beta)^9}$

**Sol** $= \dfrac{\{\cos(-\alpha)+i\sin(-\alpha)\}^{11}}{(\cos\beta+i\sin\beta)^9}$ $\quad$ ∵ To make $\cos\theta+i\sin\theta$ i.e + ive sign.

$= [(\cos\alpha+i\sin\alpha)^{-1}]^{11}[\cos\beta+i\sin\beta]^{-9}$

$$= (\cos\alpha + i\sin\alpha)^{-11}(\cos\beta + i\sin\beta)^{-9}$$

$$= [\cos(-11\alpha) + i\sin(-11\alpha)][\cos(-9\beta) + i\sin(-9\beta)]$$

$$= (\cos(-11\alpha)\cos(-9\beta) - \sin(-11\alpha)\sin(-9\beta)) + i\begin{pmatrix}\cos(-11\alpha)\sin(-9\beta) \\ +\sin(-11\alpha)\cos(-9\beta)\end{pmatrix}$$

$$= \cos(-11\alpha - 9\beta) + i\sin(-11\alpha - 9\beta)$$

$$= \text{Cis}(-11\alpha - 9\beta) = \text{Cis}(-(11\alpha + 9\beta)) \quad \text{Ans.}$$

x———x

**Part-(iii)** $\dfrac{(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)}{(\cos\gamma + i\sin\gamma)(\cos\delta + i\sin\delta)}$

**Sol**

$$= \frac{(\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta)}{(\cos\gamma\cos\delta - \sin\gamma\sin\delta) + i(\sin\gamma\cos\delta + \cos\gamma\sin\delta)}$$

$$= \frac{\cos(\alpha+\beta) + i\sin(\alpha+\beta)}{\cos(\gamma+\delta) + i\sin(\gamma+\delta)}$$

$$= [\cos(\alpha+\beta) + i\sin(\alpha+\beta)][\cos(\gamma+\delta) + i\sin(\gamma+\delta)]^{-1}$$

$$= [\cos(\alpha+\beta) + i\sin(\alpha+\beta)][\cos(-\gamma-\delta) + i\sin(-\gamma-\delta)]$$

$$= [\cos(\alpha+\beta)\cos(-\gamma-\delta) - \sin(\alpha+\beta)\sin(-\gamma-\delta)]$$
$$+ i[\sin(\alpha+\beta)\cos(-\gamma-\delta) + \sin(-\gamma-\delta)\cos(\alpha+\beta)]$$

$$= \cos(\alpha+\beta-\gamma-\delta) + i\sin(\alpha+\beta-\gamma-\delta)$$

$$= \text{Cis}(\alpha+\beta-\gamma-\delta)$$

x———x

**Part-(iv)** $\left(3\ \text{cis}\frac{\pi}{6}\right)^7 / \left(4\ \text{cis}\frac{\pi}{3}\right)^6$

**Sol:** $= 3^7\left(\text{cis}\frac{\pi}{6}\right)^7 / 4^6\left(\text{cis}\frac{\pi}{3}\right)^6$

$$= \frac{3^7}{4^6} \cdot \frac{\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^7}{\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^6}$$

1.2-5

$$= \frac{3^7}{4^6} \cdot \left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{-6}$$

$$= \frac{3^7}{4^6} \cdot \left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)\left(\cos\left(-6\cdot\frac{\pi}{3}\right) + i\sin\left(-6\cdot\frac{\pi}{3}\right)\right)$$

$$= \frac{3^7}{4^6}\left[\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right]\left[\cos(-2\pi) + i\sin(-2\pi)\right]$$

$$= \frac{3^7}{4^6}\left[\cos\left(\frac{7\pi}{6} - 2\pi\right) + i\sin\left(\frac{7\pi}{6} - 2\pi\right)\right]$$

$\because \cos\frac{7\pi}{6}\cos(-2\pi) - \sin\frac{7\pi}{6}\sin(-2\pi) = \cos\left(\frac{7\pi}{6} + (-2\pi)\right)$

$$= \frac{3^7}{4^6}\ \mathrm{Cis}\left(-\frac{5\pi}{6}\right) \quad \text{Ans.}$$

x———————x

**Q.3 (i)** Prove that $\left[(\cos\theta - \cos\phi) + i(\sin\theta - \sin\phi)\right]^n + \left[(\cos\theta - \cos\phi) - i(\sin\theta - \sin\phi)\right]^n$

$$= 2^{n+1}\sin^n\left(\frac{\theta-\phi}{2}\right)\cos n\left(\frac{\theta+\phi+\pi}{2}\right)$$

**Sol:** L.H.S. $\left[(\cos\theta - \cos\phi) + i(\sin\theta - \sin\phi)\right]^n + \left[(\cos\theta - \cos\phi) - i(\sin\theta - \sin\phi)\right]^n$

using formulas

$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

and $\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$ } we get

$$= \left[\left(-2\sin\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}\right) + i\left(2\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}\right)\right]^n$$
$$+ \left[\left(-2\sin\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}\right) - i\left(2\cos\frac{\theta+\phi}{2}\cdot\sin\frac{\theta-\phi}{2}\right)\right]^n$$

$$= 2^n\sin^n\left(\frac{\theta-\phi}{2}\right)\left[-\sin\frac{\theta+\phi}{2} + i\cos\frac{\theta+\phi}{2}\right]^n$$
$$+ 2^n\sin^n\left(\frac{\theta-\phi}{2}\right)\left[-\sin\frac{\theta+\phi}{2} - i\cos\frac{\theta+\phi}{2}\right]^n$$

$\because \cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta$ and $\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$

$$= 2^n\sin^n\left(\frac{\theta-\phi}{2}\right)\left[\left\{\cos\left(\frac{\pi}{2} + \frac{\theta+\phi}{2}\right) + i\sin\left(\frac{\pi}{2} + \frac{\theta+\phi}{2}\right)\right\}^n\right.$$
$$\left. + \left\{\cos\left(\frac{\pi}{2} + \frac{\theta+\phi}{2}\right) - i\sin\left(\frac{\pi}{2} + \frac{\theta+\phi}{2}\right)\right\}^n\right]$$

$$= 2^n \sin^n\left(\frac{\theta-\phi}{2}\right)\left[\left\{\cos n\left(\frac{\pi+\theta+\phi}{2}\right)+i\sin n\left(\frac{\pi+\theta+\phi}{2}\right)\right\}\right.$$
$$\left. + \left\{\cos n\left(\frac{\pi+\theta+\phi}{2}\right)-i\sin n\left(\frac{\pi+\theta+\phi}{2}\right)\right\}\right]$$

$$= 2^n \sin^n\left(\frac{\theta-\phi}{2}\right)\left[\begin{array}{l}\cos n\left(\frac{\pi+\theta+\phi}{2}\right) + \cancel{i \sin n\left(\frac{\pi+\theta+\phi}{2}\right)} \\ \cos n\left(\frac{\pi+\theta+\phi}{2}\right) - \cancel{i \sin n\left(\frac{\pi+\theta+\phi}{2}\right)}\end{array}\right]$$

$$= 2^n \sin^n\left(\frac{\theta-\phi}{2}\right) \, 2\cos n\left(\frac{\pi+\theta+\phi}{2}\right)$$

$$= 2^{n+1} \sin^n\left(\frac{\theta-\phi}{2}\right)\cos n\left(\frac{\pi+\theta+\phi}{2}\right) = RHS$$

x———————————x

**P. (ii)** $\left(\dfrac{1+\sin x+i\cos x}{1+\sin x-i\cos x}\right)^n = \cos n\left(\frac{\pi}{2}-x\right)+i\sin n\left(\frac{\pi}{2}-x\right)$

**Sol:** $\underline{L.H.S}$ $\left(\dfrac{1+\sin x+i\cos x}{1+\sin x-i\cos x}\right)^n$

$$= \left(\frac{(\sin^2 x+\cos^2 x)+(\sin x+i\cos x)}{1+\sin x-i\cos x}\right)^n$$

$$= \left[\frac{(\sin x+i\cos x)(\sin x-i\cos x)+(\sin x+i\cos x)}{1+\sin x-i\cos x}\right]^n$$

$$= \left[\frac{(\sin x+i\cos x)\left(\cancel{\sin x-i\cos x+1}\right)}{\left(\cancel{1+\sin x-i\cos x}\right)}\right]^n$$

$$= (\sin x+i\cos x)^n$$

$$\boxed{\begin{array}{l}\because \cos\left(\frac{\pi}{2}-x\right) = \sin x \\ \text{and} \\ \sin\left(\frac{\pi}{2}-x\right) = \cos x\end{array}}$$

$$= \left[\cos\left(\frac{\pi}{2}-x\right)+i\sin\left(\frac{\pi}{2}-x\right)\right]^n$$

applying Demoivre's th.

$$= \cos n\left(\frac{\pi}{2}-x\right)+i\sin n\left(\frac{\pi}{2}-x\right)$$

$$= R.H.S$$

**Q.4** $2\cos\theta = x + \frac{1}{x}$ ; $2\cos\phi = y + \frac{1}{y}$ , $2\cos\psi = z + \frac{1}{z}$ ,
then prove that  1.2-7

**Part-(i)** $2\cos(\theta+\phi+\psi) = xyz + \frac{1}{xyz}$

**PROOF** we have $2\cos\theta = x + \frac{1}{x}$ , $\Rightarrow x = \cos\theta + i\sin\theta$

$2\cos\phi = y + \frac{1}{y} \Rightarrow y = \cos\phi + i\sin\phi$

and. $2\cos\psi = z + \frac{1}{z} \Rightarrow z = \cos\psi + i\sin\psi$

Then $x\cdot y\cdot z = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)(\cos\psi + i\sin\psi)$

$= [(\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \sin\phi\cdot\cos\theta)][\cos\psi + i\sin\psi]$

$= [\cos(\theta+\phi) + i\sin(\theta+\phi)][\cos\psi + i\sin\psi]$

$= (\cos(\theta+\phi)\cos\psi - \sin(\theta+\phi)\sin\psi) + i(\sin(\theta+\phi)\cos\psi + \cos(\theta+\phi)\sin\psi)$

$x\cdot y\cdot z = \cos(\theta+\phi+\psi) + i\sin(\theta+\phi+\psi) \longrightarrow (1)$

Similarly $\frac{1}{x\cdot y\cdot z} = \frac{1}{(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)(\cos\psi + i\sin\psi)}$

$= \frac{1}{\cos(\theta+\phi+\psi) + i\sin(\theta+\phi+\psi)}$

$= [\cos(\theta+\phi+\psi) + i\sin(\theta+\phi+\psi)]^{-1}$

$\frac{1}{xyz} = \cos(\theta+\phi+\psi) - i\sin(\theta+\phi+\psi \longrightarrow (2)$

Add Eqns (1) and (2), we get

$xyz + \frac{1}{xyz} = 2\cos(\theta+\phi+\psi)$  Proved.

×——————×

**Part-(ii)** $2\cos(m\theta + n\phi) = x^m y^n + \frac{1}{x^m y^n}$

**Sol:** we are given that $2\cos\theta = x + \frac{1}{x}$

it is because if $x = \cos\theta + i\sin\theta$

1.2—8

and $2\cos\phi = y + \frac{1}{y} \Rightarrow y = \cos\phi + i\sin\phi$

So $x^m = (\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$

and $y^n = (\cos\phi + i\sin\phi)^n = \cos n\phi + i\sin n\phi$

then $x^m y^n = (\cos m\theta + i\sin m\theta)(\cos n\phi + i\sin n\phi)$

$= (\cos m\theta\cos n\phi - \sin m\theta\sin n\phi) + i(\sin m\theta\cos n\phi + \sin n\phi\cos m\theta)$

$$x^m y^n = \cos(m\theta + n\phi) + i\sin(m\theta + n\phi) \longrightarrow ①$$

and $\frac{1}{x^m y^n} = \frac{1}{(\cos m\theta + i\sin m\theta)(\cos n\phi + i\sin n\phi)}$

$= \frac{1}{\cos(m\theta + n\phi) + i\sin(m\theta + n\phi)}$

$= [\cos(m\theta + n\phi) + i\sin(m\theta + n\phi)]^{-1}$

$$\frac{1}{x^m y^n} = \cos(m\theta + n\phi) - i\sin(m\theta + n\phi) \longrightarrow ②$$

Add equations ① and ②, we get

$$x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\phi)$$

Proved

x————————x

Q.5(i) Find Three Cube roots of '$8i$'.

Sol Let $z^3 = 8i = 8(0 + i)$

or $z^3 = 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$\{x = 0, y = 1, r = \sqrt{0+1} = 1$
$\theta = \sin^{-1} 1 \} \theta = \frac{\pi}{2}$
1st and 2nd

or $z^3 = 2^3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$= 2^3\left(\cos\left(\frac{\pi}{2} + 2k\pi\right) + i\sin\left(\frac{\pi}{2} + 2k\pi\right)\right)$, where $k \in Z$

1·2-9

$$\Rightarrow Z_k = 2\left[\cos\left(2\pi k + \frac{\pi}{2}\right) + i \sin\left(2\pi k + \frac{\pi}{2}\right)\right]^{\frac{1}{3}}$$

where $k = 0, 1, 2$.

$$\therefore Z_k = 2\left[\cos\left(\frac{4k\pi + \pi}{6}\right) + i \sin\left(\frac{4k\pi + \pi}{6}\right)\right]$$

So put $k = 0, 1, 2$, then required three roots are given by.

for $k=0$, $Z_0 = 2\left[\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right] = 2\left\{\frac{\sqrt{3}}{2} + \frac{i \cdot 1}{2}\right\} \Rightarrow \boxed{\sqrt{3} + i = Z_0}$

for $k=1$, $Z_1 = 2\left[\cos\left(\frac{4\pi+\pi}{6}\right) + i \sin\left(\frac{4\pi+\pi}{6}\right)\right] = 2\left\{\cos\frac{5\pi}{6} + i \sin\frac{5\pi}{6}\right\}$

$$= 2\left[-\frac{\sqrt{3}}{2} + \frac{i}{2}\right] \Rightarrow \boxed{Z_1 = -\sqrt{3} + i}$$

and 3rd root is obtained by $k=2$, we get

$$Z_2 = 2\left[\cos\left(\frac{8\pi+\pi}{6}\right) + i \sin\left(\frac{8\pi+\pi}{6}\right)\right] = 2\left[\cos\frac{3\pi}{2} + i \sin\frac{3\pi}{2}\right]$$

$$= 2\left[\cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right)\right] = 2(0 - i) \Rightarrow \boxed{Z_2 = -2i}$$

x ———— x

**Part-(ii)** Find four fourth roots of each of the following complex number.

(a) $-16i$, (b) $64$, (c) $-2\sqrt{3} + 2i$

**(a)** Since we have find four fourth roots of $-16i$

So put $z^4 = -16i = 16(0 - i)$

$$= 2^4\left[\cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right)\right]$$

$x=0$, $y=-1$, $r = \sqrt{0^2 + (-1)^2} = 1$

$\cos\theta = \frac{x}{r} = \frac{0}{1} = 0 \Rightarrow \theta = \cos^{-1} 0$

$\sin\theta = \frac{y}{r} = \frac{-1}{1} = -1 \Rightarrow \theta = \cos^{-1}(-1)$ $\Rightarrow \theta = -\frac{\pi}{2}$

(Angle lies in IV^th Q ∴ $-\theta$ Hence $-\frac{\pi}{2}$)

$$z^4 = 2^4\left[\cos\left(2\pi k - \frac{\pi}{2}\right) + i \sin\left(2k\pi - \frac{\pi}{2}\right)\right]$$

So fourth root of $-16i$ is

$$Z_k = 2\left[\cos\left(2\pi k - \frac{\pi}{2}\right) + i \sin\left(2k\pi - \frac{\pi}{2}\right)\right]^{\frac{1}{4}}$$

where $k = 0, 1, 2, 3$

$$= 2\left[\cos\frac{1}{4}\left(\frac{4k\pi - \pi}{2}\right) + i \sin\left(\frac{4k\pi - \pi}{2}\right)\frac{1}{4}\right]$$

$$Z_k = 2\left[\cos\left(\frac{4k\pi - \pi}{8}\right) + i \sin\left(\frac{4k\pi - \pi}{8}\right)\right], \quad k = 0, 1, 2, 3 \longrightarrow ①$$

So required four roots can be obtained by putting $K = 0, 1, 2, 3$ in ①, we get $\boxed{1\cdot2-10}$

$$Z_0 = 2\left[\cos\left(\frac{0-\pi}{8}\right) + i\sin\left(\frac{0-\pi}{8}\right)\right] = 2\left(\cos\left(\frac{-\pi}{8}\right) + i\sin\left(\frac{-\pi}{8}\right)\right)$$

or $\boxed{Z_0 = \, cis\left(\frac{-\pi}{8}\right)}$

for $K=1$, $Z_1 = 2\left[\cos\left(\frac{4\pi-\pi}{8}\right) + i\sin\left(\frac{4\pi-\pi}{8}\right)\right] = 2 \ cis\left(\frac{3\pi}{8}\right)$

for $K=2$, $Z_2 = 2\left[\cos\left(\frac{8\pi-\pi}{8}\right) + i\sin\left(\frac{8\pi-\pi}{8}\right)\right] = 2 \ cis\left(\frac{7\pi}{8}\right)$

for $K=3$, $Z_3 = 2\left[\cos\left(\frac{12\pi-\pi}{8}\right) + i\sin\left(\frac{12\pi-\pi}{8}\right)\right] = 2 \ cis\left(\frac{11\pi}{8}\right)$ $\because \frac{11\pi}{8} - 2\pi = -\frac{5\pi}{8}$

$= 2\cdot cis\left(-\frac{5\pi}{8}\right)$

x———————x

**(b)** let $Z^4 = 64 = 64(1+0i) = 64(\cos 0 + i\sin 0)$

$$= 64(\cos(2\pi K + 0) + i\sin(2\pi K + 0))$$

$$Z^4 = 64[\cos 2\pi K + i\sin 2\pi K]$$

So four roots of 64 are

$$Z_k = (64)^{\frac{1}{4}}[\cos 2\pi K + i\sin 2\pi K]^{\frac{1}{4}}, \text{ where } k = 0,1,2,3$$

$$= (16\times 4)^{\frac{1}{4}}\left[\cos\frac{2\pi K}{4} + i\sin\frac{2\pi K}{4}\right]$$

$$= (2^4 \cdot 2^2)^{\frac{1}{4}}\left(\cos\frac{\pi K}{2} + i\sin\frac{\pi K}{2}\right]$$

$$Z_k = 2\sqrt{2}\left[\cos\frac{\pi K}{2} + i\sin\frac{\pi K}{2}\right]$$

for first root, put $k=0$, $\Rightarrow Z_0 = 2\sqrt{2}\,[\cos 0 + i\sin 0] = 2\sqrt{2}(1+i0)$

$\boxed{Z_0 = 2\sqrt{2}}$

put $K=1$, $Z_1 = 2\sqrt{2}\left\{\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right] = 2\sqrt{2}\,[0+i] \Rightarrow \boxed{2\sqrt{2}\,i = Z_1}$

put $K=2$, $Z_2 = 2\sqrt{2}\left(\cos\pi + i\sin\pi\right] = 2\sqrt{2}\,[-1+0i] \Rightarrow \boxed{Z_2 = -2\sqrt{2}}$

put $K=3$, $Z_3 = 2\sqrt{2}\left[\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right] = 2\sqrt{2}\,(0-i]$

$\Rightarrow \boxed{Z_3 = -2\sqrt{2}\,i}$

(C) Let $Z^4 = -2\sqrt{3} + 2i$   $r = |z| = \sqrt{(2\sqrt{3})^2 + 2^2} = \sqrt{12+4}$

$x = -2\sqrt{3}$, $y = 2$   $r = 4 = |z|$   $= 4\left[-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right]$   $\boxed{1\text{-}2\text{-}11}$

$\left.\begin{array}{l}\cos\theta = \frac{x}{r} = -\frac{2\sqrt{3}}{4} = -\frac{\sqrt{3}}{2} \\ \sin\theta = \frac{y}{r} = \frac{2}{4} = \frac{1}{2}\end{array}\right] \Rightarrow \theta = \frac{5\pi}{6}$   $= 4\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right]$

$\left(\begin{array}{l}\frac{x-}{y+}\ \text{2nd Q.} \therefore \pi-\theta \\ \text{So } \pi - \frac{\pi}{6} = \boxed{\frac{5\pi}{6}}\end{array}\right)$   $z^4 = 4\left[\cos\left(\frac{5\pi}{6} + 2k\pi\right) + i\sin\left(\frac{5\pi}{6} + 2k\pi\right)\right]$

$\Rightarrow z^4 = (4)\left[\cos\left(2k\pi + \frac{5\pi}{6}\right) + i\sin\left(2k\pi + \frac{5\pi}{6}\right)\right]$

$\therefore Z_k = (4)^{\frac{1}{4}}\left[\cos\left(\frac{12k\pi + 5\pi}{6}\right) + i\sin\left(\frac{12k\pi + 5\pi}{6}\right)\right]^{\frac{1}{4}}$ where $k = 0, 1, 2, 3$

$\Rightarrow Z_k = (2^2)^{\frac{1}{4}}\left[\cos\left(\frac{12k\pi + 5\pi}{24}\right) + i\sin\left(\frac{12k\pi + 5\pi}{24}\right)\right]$

$= 2^{\frac{1}{2}}\,\text{cis}\left(\frac{k\pi}{2} + \frac{5\pi}{24}\right)$ where $k = 0, 1, 2, 3$

put $k = 0$, we get

$Z_0 = \sqrt{2}\left[\cos\frac{5\pi}{24} + i\sin\frac{5\pi}{24}\right] = \sqrt{2}\ \text{Cis}\ \frac{5\pi}{24}$

for $k = 1$, $Z_1 = \sqrt{2}\left[\cos\frac{17\pi}{24} + i\sin\frac{17\pi}{24}\right] = \sqrt{2}\ \text{Cis}\ \frac{17\pi}{24}$

for $K = 2$, $Z_2 = \sqrt{2}\left[\cos\frac{29\pi}{24} + i\sin\frac{29\pi}{24}\right] = \sqrt{2}\ \text{Cis}\left(-\frac{19\pi}{24}\right)$   $\frac{29\pi}{24} - 2\pi = -\frac{19\pi}{24}$

for $K = 3$, $Z_3 = \sqrt{2}\left[\cos\frac{41\pi}{24} + i\sin\frac{41\pi}{24}\right] = \sqrt{2}\ \text{Cis}\left(-\frac{7\pi}{24}\right)$   $\frac{41\pi}{24} - 2\pi = -\frac{7\pi}{24}$

x —————— x

**Q.6** Find six 6th roots of (a) $-1$, (b) $1+i$

**Sol** (a) Let $Z^6 = -1 = (-1 + 0i)$   $r = |z| = \sqrt{(-1)^2 + 0^2} = +1$

$\left.\begin{array}{l}\cos\theta = \frac{x}{r} = \frac{-1}{1} = -1 \Rightarrow \theta = \cos^{-1}(-1) \\ \sin\theta = \frac{y}{r} = \frac{0}{1} = 0 \Rightarrow \theta = \sin^{-1}(0)\end{array}\right\}\ \theta = \pi$   $= 1[\cos\pi + i\sin\pi]$

$\left(\frac{x-}{y+}\ \text{So QII}\ \therefore \pi - \theta \text{ or } \pi - 0 = \boxed{\pi}\right)$   $z^6 = \left[\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)\right]$

So six 6th roots of '$-1$' are given by

$Z_k = \left(\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)\right)^{\frac{1}{6}}$

or $Z_k = \cos\left(\frac{2k\pi + \pi}{6}\right) + i\sin\left(\frac{2k\pi + \pi}{6}\right)$; where $k = 0, 1, 2, 3, 4, 5$

So for $k = 0$, $Z_0 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{i}{2}$

for $k=1$, $Z_1 = \cos\left(\frac{2\pi+\pi}{6}\right) + i\sin\left(\frac{2\pi+\pi}{6}\right) = \mathrm{Cis}\frac{\pi}{2}$

$\Rightarrow \boxed{Z_1 = 0 + i}$ 1.2-12

for $k=2$, $Z_2 = \cos\left(\frac{4\pi+\pi}{6}\right) + i\sin\left(\frac{4\pi+\pi}{6}\right) = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$

$\Rightarrow \boxed{Z_2 = -\frac{\sqrt{3}}{2} + \frac{i}{2}}$ $\left(\because \cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2},\ \sin\frac{5\pi}{6} = \frac{1}{2}\right)$

for $k=3$, $Z_3 = \cos\left(\frac{6\pi+\pi}{6}\right) + i\sin\left(\frac{6\pi+\pi}{6}\right) = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}$

$\frac{7\pi}{6} - 2\pi = -\frac{5\pi}{6}$ $= \cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)$

$= \cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6} = \boxed{-\frac{\sqrt{3}}{2} - \frac{i}{2}}$

for $k=4$, $Z_4 = \cos\left(\frac{8\pi+\pi}{6}\right) + i\sin\left(\frac{8\pi+\pi}{6}\right)$

$= \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = \mathrm{Cis}\left(-\frac{\pi}{2}\right)$ $\frac{3\pi}{2} - 2\pi = -\frac{\pi}{2}$

or $\boxed{Z_4 = 0 - i}$

for $k=5$, $Z_5 = \cos\left(\frac{10\pi+\pi}{6}\right) + i\sin\left(\frac{10\pi+\pi}{6}\right) = \cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}$

$Z_5 = \cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)$

or $\frac{11\pi}{6} - 2\pi = -\frac{\pi}{6}$ $\Rightarrow \boxed{Z_5 = \frac{\sqrt{3}}{2} - \frac{i}{2}}$

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**(b)** let $Z^6 = 1+i$ $\quad |Z| = |1+i| = \sqrt{1+1} = \sqrt{2} = r$

$\theta = \tan^{-1}\frac{y}{x} = \frac{1}{1} = 1 \Rightarrow \boxed{\theta = \frac{\pi}{4}}$

OR $\cos\theta = \frac{x}{r} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

$\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

$x$+ive, $y$+ive so Ist Q. $\therefore \theta$

$= \sqrt{2}\left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right] = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$

$Z^6 = \sqrt{2}\left[\cos\left(2k\pi+\frac{\pi}{4}\right) + i\sin\left(2k\pi+\frac{\pi}{4}\right)\right]$

So six 6th roots of $1+i$ are

$Z_k = \left[(2)^{\frac{1}{2}}\right]^{\frac{1}{6}}\left[\cos\left(\frac{8\pi k+\pi}{4}\right) + i\sin\left(\frac{8k\pi+\pi}{4}\right)\right]^{\frac{1}{6}}$

or $Z_k = (2)^{\frac{1}{12}}\left[\cos\left(\frac{8\pi k+\pi}{24}\right) + i\sin\left(\frac{8\pi k+\pi}{24}\right)\right]$ where $k = 0,1,2,3,4,5$

for $k=0$, $Z_0 = (2)^{\frac{1}{12}}\left[\cos\frac{\pi}{24} + i\sin\frac{\pi}{24}\right] = 2^{\frac{1}{12}} cis \frac{\pi}{24}$

for $k=1$, $Z_1 = (2)^{\frac{1}{12}}\left[\cos\frac{9\pi}{24} + i\sin\frac{9\pi}{24}\right] = 2^{\frac{1}{12}} cis \frac{3\pi}{8}$

for $k=2$, $Z_2 = (2)^{\frac{1}{12}}\left[\cos\frac{17\pi}{24} + i\sin\frac{17\pi}{24}\right] = 2^{\frac{1}{12}} cis \frac{17\pi}{24}$

for $k=3$, $Z_3 = (2)^{\frac{1}{12}}\left[\cos\frac{25\pi}{24} + i\sin\frac{25\pi}{24}\right]$

$= (2)^{\frac{1}{12}}\left[\cos\left(\pi+\frac{\pi}{24}\right) + i\sin\left(\pi+\frac{\pi}{24}\right)\right]$

$= (2)^{\frac{1}{12}}\left[-\cos\frac{\pi}{24} - i\sin\frac{\pi}{24}\right]$

$= -(2)^{\frac{1}{12}}\left[\cos\frac{\pi}{24} + i\sin\frac{\pi}{24}\right] = -2^{\frac{1}{12}} cis \frac{\pi}{24}$

Also $\frac{25\pi}{24} - 2\pi = -\frac{23\pi}{24}$

$Z_3 = 2^{\frac{1}{12}} cis\left(-\frac{23\pi}{24}\right)$

for $k=4$, $Z_4 = (2)^{\frac{1}{12}}\left[\cos\frac{33\pi}{24} + i\sin\frac{33\pi}{24}\right]$

$= (2)^{\frac{1}{12}}\left[\cos\left(\pi+\frac{9\pi}{24}\right) + i\sin\left(\pi+\frac{9\pi}{24}\right)\right]$

$= (2)^{\frac{1}{12}}\left[\cos\left(\pi+\frac{3\pi}{8}\right) + i\sin\left(\pi+\frac{3\pi}{8}\right)\right]$

$= (2)^{\frac{1}{12}}\left[-\cos\frac{3\pi}{8} - i\sin\frac{3\pi}{8}\right]$

$= -(2)^{\frac{1}{12}}\left[\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right]$

Also $\frac{33\pi}{24} - 2\pi = -\frac{15\pi}{24}$

$Z_4 = 2^{\frac{1}{12}} cis\left(-\frac{15\pi}{24}\right)$

for $k=5$, $Z_5 = (2)^{\frac{1}{12}}\left[\cos\frac{41\pi}{24} + i\sin\frac{41\pi}{24}\right]$

$= (2)^{\frac{1}{2}}\left[\cos\left(\pi+\frac{17\pi}{24}\right) + i\sin\left(\pi+\frac{17\pi}{24}\right)\right]$

$= +(2)^{\frac{1}{2}}\left[-\cos\frac{17\pi}{24} - i\sin\frac{17\pi}{24}\right]$

$= -(2)^{\frac{1}{2}}\left[\cos\frac{17\pi}{24} + i\sin\frac{17\pi}{24}\right]$

Also $\frac{41\pi}{24} - 2\pi = -\frac{7\pi}{24}$

$Z_5 = 2^{\frac{1}{12}} cis\left(-\frac{7\pi}{24}\right)$

**Q:7** Find the squares of all the fifth roots of $\frac{1}{2}+\frac{\sqrt{3}}{2}i$

Sol

$r=\sqrt{(\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}=\sqrt{\frac{1}{4}+\frac{3}{4}}=1$

$\cos\theta=\frac{x}{r}=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{3}$

$\sin\theta=\frac{y}{r}=\frac{\sqrt{3}}{2} \Rightarrow \theta=\frac{\pi}{3}$

same so Quad I

let $z^5=\frac{1}{2}+\frac{\sqrt{3}}{2}i = 1\left[\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right]$

or $z^5=\left[\cos(2k\pi+\frac{\pi}{3})+i\sin(2k\pi+\frac{\pi}{3})\right]$

$\boxed{1\cdot 2-14}$

$$Z_k=\left[\cos\left(\frac{6k\pi+\pi}{3}\right)+i\sin\left(\frac{6k\pi+\pi}{3}\right)\right]^{1/5}$$

$$Z_k=\left[\cos\left(\frac{6k\pi+\pi}{15}\right)+i\sin\frac{6k\pi+\pi}{15}\right] \text{ where } k=0,1,2,3,4$$

Now square of all the 5th root are

$$Z_k=\left[\cos\left(\frac{6k\pi+\pi}{15}\right)+i\sin\left(\frac{6k\pi+\pi}{15}\right)\right]^2 \text{ where } k=0,1,2,3,4$$

or $$Z_k=\cos\left(\frac{12k\pi+2\pi}{15}\right)+i\sin\left(\frac{12k\pi+2\pi}{15}\right) \text{ where } k=0,1,2,3,4$$

for $k=0$, $Z_0=\cos\frac{2\pi}{15}+i\sin\frac{2\pi}{15}=\text{Cis}\,\frac{2\pi}{15}$

for $K=1$, $Z_1=\cos\frac{14\pi}{15}+i\sin\frac{14\pi}{15}=\text{Cis}\,\frac{14\pi}{15}$

for $K=2$, $Z_2=\cos\frac{26\pi}{15}+i\sin\frac{26\pi}{15}$

$\frac{26\pi}{15}-2\pi=\frac{-4\pi}{15}$

$$=\cos\left(\frac{-4\pi}{15}\right)+i\sin\left(\frac{-4\pi}{15}\right)=\text{Cis}\left(\frac{-4\pi}{15}\right)$$

for $K=3$, $Z_3=\cos\frac{38\pi}{15}+i\sin\frac{38\pi}{15}$

$\frac{38\pi}{15}-2\pi=\frac{8\pi}{15}$

$$Z_3=\cos\frac{8\pi}{15}+i\sin\frac{8\pi}{15}=\text{Cis}\left(\frac{8\pi}{15}\right)$$

$15\overline{)50}\,3$, $\frac{45}{5}$, $(3+\frac{5}{15})\pi$

$$Z_4=\cos\frac{50\pi}{15}+i\sin\frac{50\pi}{15}$$

for $K=4$, $Z_4=\cos\frac{20\pi}{15}+i\sin\frac{20\pi}{15}$

$\frac{50\pi}{15}-2\pi=\frac{20\pi}{15}$

$\frac{20\pi}{15}-2\pi=-\frac{10\pi}{15}=\frac{-2\pi}{3}$

$$=\cos\left(\frac{-2\pi}{3}\right)+i\sin\left(\frac{-2\pi}{3}\right)=\text{Cis}\left(-\frac{2\pi}{3}\right)$$

**Q.8 (i)** Solve the equation $x^7+1=0$ 

$\boxed{1 \cdot 2 - 15}$

**Sol** We have $x^7+1=0$

$$\Rightarrow x^7 = -1 = -1+0i = 1(\cos\pi + i\sin\pi)$$

or $x^7 = \cos(\pi+2k\pi) + i\sin(\pi+2k\pi)$

$$\Rightarrow x = \left[\cos(\pi+2k\pi) + i\sin(\pi+2k\pi)\right]^{\frac{1}{7}}$$

So Seven 7th roots of $-1$ are

$$x_k = \cos\left(\frac{\pi+2k\pi}{7}\right) + i\sin\left(\frac{\pi+2k\pi}{7}\right)$$

Where $k = 0, 1, 2, 3, 4, 5, 6$

for $k=0$, $x_0 = \cos\frac{\pi}{7} + i\sin\frac{\pi}{7} = \text{cis}\,\frac{\pi}{7}$

$\frac{9\pi}{7} = (1+\frac{2}{7})\pi$ $\quad 7\overline{)9}1$ $\quad \frac{7}{2}$

$= \pi + \frac{2\pi}{7}$

for $k=1$, $x_1 = \cos\frac{3\pi}{7} + i\sin\frac{3\pi}{7} = \text{cis}\,\frac{3\pi}{7}$

for $k=2$, $x_2 = \cos\frac{5\pi}{7} + i\sin\frac{5\pi}{7} = \text{cis}\,\frac{5\pi}{7}$

for $k=3$, $x_3 = \cos\pi + i\sin\pi = -1+0i = -1$

for $k=4$, $x_4 = \cos\frac{9\pi}{7} + i\sin\frac{9\pi}{7}$

$\frac{9\pi}{7} - 2\pi = -\frac{5\pi}{7}$

$$= \cos\left(-\frac{5\pi}{7}\right) + i\sin\left(-\frac{5\pi}{7}\right)$$

$$x_4 = \text{cis}\left(-\frac{5\pi}{7}\right)$$

for $k=5$ $\quad x_5 = \cos\left(\frac{11\pi}{7}\right) + i\sin\left(\frac{11\pi}{7}\right)$

$\frac{11\pi}{7} - 2\pi = -\frac{3\pi}{7}$

$$= \cos\left(-\frac{3\pi}{7}\right) + i\sin\left(-\frac{3\pi}{7}\right)$$

$$= \text{cis}\left(-\frac{3\pi}{7}\right)$$

for $k=6$, $x_6 = \cos\frac{13\pi}{7} + i\sin\frac{13\pi}{7}$

$\frac{13\pi}{7} - 2\pi = -\frac{\pi}{7}$

$$= \cos\left(-\frac{\pi}{7}\right) + i\sin\left(-\frac{\pi}{7}\right) = \text{cis}\left(-\frac{\pi}{7}\right)$$

**Note** we can also take values of $k = 0, \pm 1, \pm 2, \pm 3$ instead of $k = 0, 1, 2, 3, 4, 5, 6$

P-(ii) $x^7+x^4+x^3+1=0$ $\boxed{1\cdot 2-15}$

Sol $x^4[x^3+1]+1[x^3+1]=0$

$\Rightarrow (x^4+1)(x^3+1)=0$

$\Rightarrow x^4+1=0$ and $x^3+1=0$

In case of $x^4+1=0 \Rightarrow x^4=-1=-1+0i$

or $x^4=\cos\pi+i\sin\pi = \cos(2k\pi+\pi)+i\sin(2k\pi+\pi)$

$\Rightarrow x_k=\cos\left(\frac{2k\pi+\pi}{4}\right)+i\sin\left(\frac{2k\pi+\pi}{4}\right)$, where $k=0,1,2,3$

for $k=0$, $x_0=\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}$

for $k=1$, $x_1=\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}=-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}$

for $k=2$, $x_2=\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}=\cos\frac{-3\pi}{4}+i\sin\left(\frac{-3\pi}{4}\right)=-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}$

$\frac{5\pi}{4}-2\pi=-\frac{3\pi}{4}$

for $k=3$, $x_3=\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}=\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}$

$\frac{7\pi}{4}-2\pi=-\frac{\pi}{4}$

In case of $x^3+1=0 \Rightarrow x^3=-1=-1+0i$

or $x^3=\cos\pi+i\sin\pi = \cos(2k\pi+\pi)+i\sin(2k\pi+\pi)$

$\Rightarrow x_k=\cos\left(\frac{2k\pi+\pi}{3}\right)+i\sin\left(\frac{2k\pi+\pi}{3}\right)$, $k=0,1,-1$

for $k=-1$, $x_{-1}=\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)=\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}=\frac{1}{2}-\frac{\sqrt{3}}{2}i$

for $k=0$, $x_0=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}=\frac{1}{2}+\frac{\sqrt{3}}{2}i$

for $k=1$, $x_1=\cos\pi+i\sin\pi$

$=-1+0=-1$

$r = |z| = \sqrt{(1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$

1·2–17

**Q.8(iii)** $x^6 + 1 = \sqrt{3}\,i$

$$\Rightarrow x^6 = -1 + \sqrt{3}\,i$$

or $x^6 = 2\left[-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right]$ ($\times$ and $\div$ by 2, we get)

2nd quad

For finding $\theta$, find $\theta = \cos^{-1}(-0.5)$

or $x^6 = 2\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]$

$$= 2\left[\cos\left(2k\pi + \frac{2\pi}{3}\right) + i\sin\left(2k\pi + \frac{2\pi}{3}\right)\right]$$

$$\Rightarrow x = (2)^{\frac{1}{6}}\left[\cos\left(2k\pi + \frac{2\pi}{3}\right) + i\sin\left(2k\pi + \frac{2\pi}{3}\right)\right]^{1/6}$$

or $x = (2)^{\frac{1}{6}}\left[\cos\left(\frac{6k\pi + 2\pi}{18}\right) + i\sin\left(\frac{6k\pi + 2\pi}{18}\right)\right]$

where $k = 0, 1, 2, 3, 4, 5$

for $k = 0$, $x_0 = (2)^{\frac{1}{6}}\left[\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right]$

for $k = 1$, $x_1 = (2)^{\frac{1}{6}}\left[\cos\frac{8\pi}{18} + i\sin\frac{8\pi}{18}\right] = (2)^{\frac{1}{6}}\,\mathrm{cis}\left(\frac{4\pi}{9}\right)$

for $k = 2$, $x_2 = (2)^{\frac{1}{6}}\left[\cos\frac{7\pi}{9} + i\sin\frac{7\pi}{9}\right] = (2)^{\frac{1}{6}}\,\mathrm{cis}\left(\frac{7\pi}{9}\right)$

for $k = 3$, $x_3 = (2)^{\frac{1}{6}}\left[\cos\frac{20\pi}{18} + i\sin\frac{20\pi}{18}\right]$

$\frac{20\pi}{18} - 2\pi = \frac{-16\pi}{18}$

$$= (2)^{\frac{1}{6}}\left[\cos\left(\frac{-16\pi}{18}\right) + i\sin\left(\frac{-16\pi}{18}\right)\right] = (2)^{\frac{1}{6}}\,\mathrm{cis}\left(\frac{-8\pi}{9}\right)$$

for $k = 4$, $x_4 = (2)^{\frac{1}{6}}\left[\cos\frac{13\pi}{9} + i\sin\frac{13\pi}{9}\right] = (2)^{\frac{1}{6}}\,\mathrm{cis}\left(\frac{5\pi}{9}\right)$

$\frac{13\pi}{9} - 2\pi = \frac{-5\pi}{9}$

for $k = 4$, $x_5 = (2)^{\frac{1}{6}}\left[\cos\frac{16\pi}{9} + i\sin\frac{16\pi}{9}\right]$

$\frac{16\pi}{9} - 2\pi = \frac{-2\pi}{9}$

$$= (2)^{\frac{1}{6}}\left[\cos\left(\frac{-2\pi}{9}\right) + i\sin\left(\frac{-2\pi}{9}\right)\right] = (2)^{\frac{1}{6}}\,\mathrm{cis}\left(\frac{-2\pi}{9}\right)$$

---

**Q.9** Solve the equation $x^{12} - 1 = 0$ and find which of its roots satisfy the equation $x^4 + x^2 + 1 = 0$

**Sol:** $x^{12} - 1 = 0$ gives

$x^{12} = 1 = 1 + 0i = \cos 0 + i\sin 0$

or $x^{12} = \cos(0 + 2k\pi) + i\sin(0 + 2k\pi)$

Easy

(21)

12-18

2nd Method: Now $x^4+x^2+1=0$

$x^4+x^2+1=0$
$x^4+x^2+x^2-x^2+1=0$
$x^4+2x^2+1-x^2=0$
$(x^2+1)^2-x^2=0$
$(x^2+1+x)(x^2+1-x)=0$

Either
$x^2+1+x=0$
$x=\frac{-1\pm\sqrt{1-4}}{2}$
$x=\frac{-1\pm i\sqrt{3}}{2}$

OR
$x^2+1-x=0$
$x=\frac{1\pm\sqrt{1-4}}{2}$
$=\frac{1\pm i\sqrt{3}}{2}$

Put $x^2=y$

$y^2+y+1=0$

$y=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm i\sqrt{3}}{2}$

x and by 2 to complete square

$x^2=\frac{-2\pm2\sqrt{-3}}{4}$

$=\frac{-2\pm2\cdot1\cdot\sqrt{-3}}{4}$

$=\frac{(1-3)\pm2\cdot1\cdot\sqrt{-3}}{4}$

$x^2=\frac{1^2+(\sqrt{-3})^2\pm2\cdot1\cdot\sqrt{-3}}{4}$

$x^2=\left(\frac{1\pm\sqrt{-3}}{2}\right)^2$

$x=\pm\left(\frac{1\pm\sqrt{-3}}{2}\right)$

$\Rightarrow x=\frac{1\pm\sqrt{3}i}{2},\ \frac{-1\pm\sqrt{3}i}{2}$

$x=\frac{1+\sqrt{3}i}{2},\ \frac{1-\sqrt{3}i}{2},\ \frac{-1+\sqrt{3}i}{2},\ \frac{-1-\sqrt{3}i}{2}$

We see that in twelve 12th roots of $x^{12}=-1$

$x_2$, $x_4$, $x_8$ and $x_{10}$ are satisfy the roots of equation $x^4+x^2+1=0$

x———————x

**Q. 10** Expand the following in series of Sines or Cosines of multiples of $\theta$

$(a+b)^n=a^n+na^{n-1}\cdot b+\frac{n(n-1)}{\lfloor 2}a^{n-2}b^2+\ldots\ldots b^n$

(i) $\cos^4\theta$

Sol: let $x=\cos\theta+i\sin\theta$, then $\frac{1}{x}=\frac{1}{\cos\theta+i\sin\theta}$
$=(\cos\theta+i\sin\theta)^{-1}$
$\frac{1}{x}=\cos\theta-i\sin\theta$

$\Rightarrow x+\frac{1}{x}=2\cos\theta$

So $(2\cos\theta)^4=\left(x+\frac{1}{x}\right)^4$

$2^4\cos^4\theta=x^4+4x^3\left(\frac{1}{x}\right)+\frac{4\cdot3}{2\cdot1}x^2\cdot\frac{1}{x^2}+\frac{4\cdot3\cdot2}{3\cdot2\cdot1}x\cdot\frac{1}{x^3}+\frac{1}{x^4}$

$=x^4+4x^2+6+\frac{4}{x^2}+\frac{1}{x^4}$

$$x^{12} = \cos 2\pi k + i \sin 2\pi k$$ $\boxed{1\cdot 2-19}$

or

$$\Rightarrow x_k = (\cos 2\pi k + i \sin 2\pi k)^{1/12}$$

$$\Rightarrow x_k = \cos \frac{\pi k}{6} + i \sin \frac{\pi k}{6}$$

where $k = 0, 1, 2, \ldots\ldots 11$

For $k=0$, $x_0 = \cos 0 + i \sin 0 = 1$

For $k=1$, $x_1 = \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} = \frac{\sqrt{3}}{2} + i\frac{1}{2} = \frac{\sqrt{3}+i}{2}$

For $k=2$, $x_2 = \cos \frac{2\pi}{6} + i \sin \frac{2\pi}{6} = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2} i = \boxed{\frac{1+\sqrt{3}i}{2}}$

For $k=3$, $x_3 = \cos \frac{3\pi}{6} + i \sin \frac{3\pi}{6} = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = 0 + i = i$

For $k=4$, $x_4 = \cos \frac{4\pi}{6} + i \sin \frac{4\pi}{6} = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = \cos\left(\frac{\pi}{3} + \frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3} + \frac{\pi}{3}\right)$

$$= \cos\frac{\pi}{3} \cdot \cos\frac{\pi}{3} - \sin\frac{\pi}{3} \sin\frac{\pi}{3} + i\left(2 \sin\frac{\pi}{3} \cos\frac{\pi}{3}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} + 2i\frac{\sqrt{3}}{2}\cdot\frac{1}{2}$$

$$= \frac{1}{4} - \frac{(\sqrt{3})^2}{4} + i\frac{\sqrt{3}}{2} = -\frac{2}{4} + \frac{\sqrt{3}i}{2} = -\frac{1}{2} + \frac{\sqrt{3}i}{2} = \boxed{\frac{-1+\sqrt{3}i}{2}}$$

$x_5 = \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{i}{2} = \frac{-\sqrt{3}+i}{2}$

$x_6 = \cos \pi + i \sin \pi = -1 + 0 i = -1$

$\because \frac{7\pi}{6} - 2\pi = -\frac{5\pi}{6}$ $x_7 = \cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6} = \cos\left(-\frac{5\pi}{6}\right) + i \sin\left(-\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} - \frac{i}{2}$

$\frac{8\pi}{6} - 2\pi = -\frac{4\pi}{6}$ $x_8 = \cos \frac{8\pi}{6} + i \sin \frac{8\pi}{6} = \cos\left(-\frac{4\pi}{6}\right) + i \sin\left(-\frac{4\pi}{6}\right) = \mathrm{cis}\left(-\frac{2\pi}{3}\right) = \boxed{\frac{-1-i\sqrt{3}}{2}}$

$\frac{9\pi}{6} - 2\pi = -\frac{3\pi}{6}$ $x_9 = \cos \frac{9\pi}{6} + i \sin \frac{9\pi}{6} = \cos\left(-\frac{3\pi}{2}\right) + i \sin\left(-\frac{3\pi}{2}\right) = 0 - i(1) = -i$

$\frac{5\pi}{3} - 2\pi = -\frac{\pi}{3}$ $x_{10} = \cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} = \cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right)$

$$= \cos\frac{\pi}{3} - i \sin\frac{\pi}{3} = \boxed{\frac{1-\sqrt{3}i}{2}}$$

$\frac{11\pi}{6} - 2\pi = -\frac{\pi}{6}$ $x_{11} = \cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6} = \cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right)$

$$= \cos\frac{\pi}{6} - i \sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{i}{2} = \frac{\sqrt{3}-i}{2}$$

$$2^4\cos^4\theta = \left(x^4+\frac{1}{x^4}\right)+4\left(x^2+\frac{1}{x^2}\right)+6$$

If $x = \cos\theta + i\sin\theta$
then $x^4 = \cos 4\theta + i\sin 4\theta$
$\frac{1}{x^4} = \frac{1}{\cos 4\theta + i\sin 4\theta}$
$\frac{1}{x^4} = \cos 4\theta - i\sin 4\theta$
$x^4 + \frac{1}{x^4} = 2\cos 4\theta$

$$= 2\cos 4\theta + 4\cdot 2\cos 2\theta + 6$$

$$2^4\cos^4\theta = 2\left[\cos 4\theta + 4\cos 2\theta + 3\right]$$

$$\cos^4\theta = \frac{1}{2^3}\left[\cos 4\theta + 4\cos 2\theta + 3\right] = \frac{1}{8}\left[\cos 4\theta + 4\cos 2\theta + 3\right]$$

× ———— ×

(ii) $\sin^4\theta$

**Sol** if $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$

so $x - \frac{1}{x} = 2i\sin\theta$, thus

$$(2i\sin\theta)^4 = \left(x-\frac{1}{x}\right)^4 = x^4 - 4x^2 + 6 - \frac{4}{x^2} + \frac{1}{x^4}$$

(∵ similar to part (i))

$$2^4 i^4 \sin^4\theta = \left(x^4+\frac{1}{x^4}\right) - 4\left(x^2+\frac{1}{x^2}\right) + 6 \qquad \boxed{1\cdot2\text{-}2\theta}$$

$$= 2\cos 4\theta - 4(2\cos 2\theta) + 6$$

$$16 i^4\sin^4\theta = 2(\cos 4\theta - 4\cos 2\theta + 3)$$

$$\sin^4\theta = \frac{2}{16}\left[\cos 4\theta - 4\cos 2\theta + 3\right]$$

$$\Rightarrow \sin^4\theta = \frac{1}{8}\left[\cos 4\theta - 4\cos 2\theta + 3\right]$$

× ———— ×

(iii) $\sin^6\theta$

**Sol.** Let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$

$$x - \frac{1}{x} = 2i\sin\theta$$

$$\Rightarrow (2i\sin\theta)^6 = \left(x-\frac{1}{x}\right)^6$$

$$(2i\sin\theta)^6 = x^6 - 6x^5\cdot\frac{1}{x} + \frac{6\cdot5}{2\cdot1}x^4\cdot\frac{1}{x^2} - \frac{6\cdot5\cdot4}{3\cdot2\cdot1}x^3\cdot\frac{1}{x^3} + \frac{6\cdot5\cdot4\cdot3}{4\cdot3\cdot2\cdot1}x^2\cdot\frac{1}{x^4}$$

$$- \frac{6\cdot5\cdot4\cdot3\cdot2}{5\cdot4\cdot3\cdot2\cdot1}\cdot x\cdot\frac{1}{x^5} + \frac{1}{x^6}$$

$$= x^6 - 6x^4 + 15x^2 - 20 + \frac{15}{x^2} - \frac{6}{x^4} + \frac{1}{x^6}$$

$$2^6 i^6 \sin^6\theta = \left(x^6+\frac{1}{x^6}\right) - 6\left(x^4+\frac{1}{x^4}\right) + 15\left(x^2+\frac{1}{x^2}\right) - 20$$

$$-2^6 \sin^6\theta = (2\cos6\theta) - 6(2\cos4\theta) + 15(2\cos2\theta) - 20$$

$$-2^6 \sin^6\theta = 2\left[\cos6\theta - 6\cos4\theta + 15\cos2\theta - 10\right]$$

$$\Rightarrow \boxed{\sin^6\theta = -\frac{1}{2^5}\left[\cos6\theta - 6\cos4\theta + 15\cos2\theta - 10\right]}$$

ans

x ———— x

1.2-21

**(iv) 10** $\cos^7\theta = ?$

**Sol:** let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$

$$\Rightarrow x + \frac{1}{x} = 2\cos\theta$$

So $\left[2\cos\theta\right]^7 = \left(x+\frac{1}{x}\right)^7$

$$2^7\cos^7\theta = x^7 + 7x^6\cdot\frac{1}{x} + \frac{7\cdot6}{2\cdot1}x^5\cdot\frac{1}{x^2} + \frac{7\cdot6\cdot5}{3\cdot2\cdot1}x^4\cdot\frac{1}{x^3} + \frac{7\cdot6\cdot5\cdot4}{4\cdot3\cdot2\cdot1}x^3\cdot\frac{1}{x^4} + \frac{7\cdot6\cdot5\cdot4\cdot3}{5\cdot4\cdot3\cdot2\cdot1}x^2\cdot\frac{1}{x^5}$$

$$+ \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2}{6\cdot5\cdot4\cdot3\cdot2\cdot1}\, x\cdot\frac{1}{x^6} + \frac{1}{x^7}$$

or $$2^7\cos^7\theta = x^7 + 7x^5 + 21x^3 + 35x + \frac{35}{x} + \frac{21}{x^3} + \frac{7}{x^5} + \frac{1}{x^7}$$

$$2^7\cos^7\theta = \left(x^7+\frac{1}{x^7}\right) + 7\left(x^5+\frac{1}{x^5}\right) + 21\left(x^3+\frac{1}{x^3}\right) + 35\left(x+\frac{1}{x}\right)$$

$$= 2\cos7\theta + 7(2\cos5\theta) + 21(2\cos3\theta) + 35(2\cos\theta)$$

$$\Rightarrow 2^7\cos^7\theta = 2\left[\cos7\theta + 7\cos5\theta + 21\cos3\theta + 35\cos\theta\right]$$

$$\Rightarrow \boxed{\cos^7\theta = \frac{1}{2^6}\left[\cos7\theta + 7\cos5\theta + 21\cos3\theta + 35\cos\theta\right]}$$

ans.

(v) Q-10 $\sin^9\theta = ?$ $\boxed{1\text{-}2\text{-}22}$

**SOL:-** let $x = \cos\theta + i\sin\theta$ then

$\frac{1}{x} = \cos\theta - i\sin\theta$

$\Rightarrow x - \frac{1}{x} = 2i\sin\theta$

$\Rightarrow (2i\sin\theta)^9 = \left(x - \frac{1}{x}\right)^9$

or $2^9 i^9 \sin^9\theta = x^9 - 9x^8\frac{1}{x} + \frac{9\cdot 8\, x^7}{2\cdot 1\cdot x^2} - \frac{9\cdot 8\cdot 7}{3\cdot 2\cdot 1}\frac{x^6}{x^3} + \frac{9\cdot 8\cdot 7\cdot 6}{4\cdot 3\cdot 2\cdot 1}\frac{x^5}{x^4}$

$- \frac{9\cdot 8\cdot 7\cdot 6\cdot 5}{5\cdot 4\cdot 3\cdot 2\cdot 1}\cdot\frac{x^4}{x^5} + \frac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}\cdot\frac{x^3}{x^6} - \frac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}\frac{x^2}{x^7}$

$+ \frac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}{8!}\frac{x}{x^8} - \frac{1}{x^9}$

$2^9 i^{8+1}\sin^9\theta = x^9 - 9x^7 + 36x^5 - 84x^3 + 126x - \frac{126}{x}$

$+ \frac{84}{x^3} - \frac{36}{x^5} + \frac{9}{x^7} - \frac{1}{x^9}$

$2^9 i\sin^9\theta = x^9 - 9x^7 + 36x^5 - 84x^3 + 126x - \frac{126}{x}$

$+ \frac{84}{x^3} - \frac{36}{x^5} + \frac{9}{x^7} - \frac{1}{x^9}$

$= \left(x^9 - \frac{1}{x^9}\right) - 9\left(x^7 - \frac{1}{x^7}\right) + 36\left(x^5 - \frac{1}{x^5}\right) - 84\left(x^3 - \frac{1}{x^3}\right)$

$+ 126\left(x - \frac{1}{x}\right)$

$= 2i\sin 9\theta - 9(2i\sin 7\theta) + 36(2i\sin 5\theta) - 84(2i\sin 3\theta)$

$+ 126(2i\sin\theta)$

$2^9 i\sin^9\theta = 2i\left[\sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta\right]$

$$\boxed{\sin^9\theta = \frac{1}{2^8}\left[\sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta\right]}$$

Ans.

(vi) Q-10 $\sin^6\theta\cos^2\theta = ?$

**SOL:-** let $x = \cos\theta + i\sin\theta$ then $\frac{1}{x} = \cos\theta - i\sin\theta$ $\Big\}$ $\therefore$ $x + \frac{1}{x} = 2\cos\theta$

and $x - \frac{1}{x} = 2i\sin\theta$

$$2^6 i^6 2^2 \sin^6\theta \cos^2\theta = \left(x-\frac{1}{x}\right)^6\left(x+\frac{1}{x}\right)^2$$

$$2^8 i^6 \sin^6\theta\cos^2\theta = \left[x-\frac{1}{x}\right]^4\left[x-\frac{1}{x}\right]^2\left[x+\frac{1}{x}\right]^2 = \left[x-\frac{1}{x}\right]^4\left[\left(x-\frac{1}{x}\right)\left(x+\frac{1}{x}\right)\right]^2$$

$$= \left[x-\frac{1}{x}\right]^4\left[x^2-\frac{1}{x^2}\right]^2$$

$$= \left[x^4 - 4x^3\cdot\frac{1}{x} + \frac{4\cdot 3}{2\cdot 1}x^2\cdot\frac{1}{x^2} - \frac{4\cdot 3\cdot 2}{3\cdot 2\cdot 1}\frac{x}{x^3} + \frac{1}{x^4}\right]\left[x^4+\frac{1}{x^4}-2\right]$$

$$= \left[x^4 - 4x^2 + 6 - \frac{4}{x^2} + \frac{1}{x^4}\right]\left[x^4+\frac{1}{x^4}-2\right]$$

$$= x^8 + 1 - 2x^4 - 4x^6 - \frac{4}{x^2} + 8x^2 + 6x^4 + \frac{6}{x^4} - 12 - 4x^2$$
$$- \frac{4}{x^6} + \frac{8}{x^2} + 1 + \frac{1}{x^8} - \frac{2}{x^4}$$

$$\boxed{i^6 = -1}$$

$$= \left(x^8+\frac{1}{x^8}\right) - 4\left(x^6+\frac{1}{x^6}\right) + \left(-2x^4+6x^4+\frac{6}{x^4}-\frac{2}{x^4}\right)$$
$$+ \left(8x^2-4x^2+\frac{8}{x^2}-\frac{4}{x^2}\right) - 10$$

$$-2^8 \sin^6\theta\cos^2\theta = \left(x^8+\frac{1}{x^2}\right) - 4\left(x^6+\frac{1}{x^6}\right) + 4\left(x^4+\frac{1}{x^4}\right) + 4\left(x^2+\frac{1}{x^2}\right) - 10$$

$$= 2\cos 8\theta - 4(2\cos 6\theta) + 4(2\cos 4\theta) + 4(2\cos 2\theta) - 10$$

$$-2^8\cdot\sin^6\theta\cos^2\theta = 2\left[\cos 8\theta - 4\cos 6\theta + 4\cos 4\theta + 4\cos 2\theta - 5\right]$$

$$\Rightarrow \sin^6\theta\cos^2\theta = -\frac{1}{2^7}\left(\cos 8\theta - 4\cos 6\theta + 4\cos 4\theta + 4\cos 2\theta - 5\right) \quad \text{Ans}$$

or
$$\boxed{\sin^6\theta\cos^2\theta = \frac{1}{2^7}\left(-\cos 8\theta + 4\cos 6\theta - 4\cos 4\theta - 4\cos 2\theta + 5\right)}$$

——x —— ans ——x——

**(vii) Q-10** $\cos^4\theta\sin^3\theta = ?$

**Sol:** Let $x = \cos\theta + i\sin\theta$
Then $\frac{1}{x} = \cos\theta - i\sin\theta$ $\Big\}\Rightarrow x+\frac{1}{x} = 2\cos\theta$ and $x-\frac{1}{x} = 2i\sin\theta$

So $(2\cos\theta)^4(2i\sin\theta)^3 = \left(x+\frac{1}{x}\right)^4\left(x-\frac{1}{x}\right)^3$

$$2^4 \cdot 2^3\, i^3 \cos^4\theta \sin^3\theta = \left(x+\frac{1}{x}\right)\left(x+\frac{1}{x}\right)^3\left(x-\frac{1}{x}\right)^3$$

$$-2^7\, i \cos^4\theta \sin^3\theta = \left(x+\frac{1}{x}\right)\left[\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)\right]^3 \qquad \boxed{1.2-24}$$

$$= \left(x+\frac{1}{x}\right)\left[x^2-\frac{1}{x^2}\right]^3$$

$$= \left(x+\frac{1}{x}\right)\left[(x^2)^3 - 3(x^2)^2\left(\frac{1}{x^2}\right) + 3x^2 \cdot \frac{1}{(x^2)^2} - \frac{1}{(x^2)^3}\right]$$

$$= \left(x+\frac{1}{x}\right)\left[x^6 - 3x^2 + \frac{3}{x^2} - \frac{1}{x^6}\right]$$

$$= x^7 - 3x^3 + \frac{3}{x} - \frac{1}{x^5} + x^5 - 3x + \frac{3}{x^3} - \frac{1}{x^7}$$

$$= \left(x^7 - \frac{1}{x^7}\right) - 3\left(x^3 - \frac{1}{x^3}\right) - 3\left(x - \frac{1}{x}\right) + \left(x^5 - \frac{1}{x^5}\right)$$

$$= 2i \sin 7\theta - 6i \sin 3\theta - 6i \sin\theta + 2i \sin 5\theta$$

$$-2^7\, i \cos^4\theta \sin^3\theta = 2i\left[\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta\right]$$

$$\boxed{\Rightarrow \cos^4\theta \sin^3\theta = -\frac{1}{2^6}\left[\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta\right]}$$

—x— Ans —x—

VII Q-10 $\cos^5\theta \sin^7\theta = ?$

Sol:- Let $x = \cos\theta + i\sin\theta$, $\frac{1}{x} = \cos\theta - i\sin\theta$ $\Big\} \Rightarrow x + \frac{1}{x} = 2\cos\theta$ and $x - \frac{1}{x} = 2i\sin\theta$

So: $(2\cos\theta)^5 (2i\sin\theta)^7 = \left(x+\frac{1}{x}\right)^5\left(x-\frac{1}{x}\right)^7$

$$-2^{12}\, i \cos^5\theta \sin^7\theta = \left(x+\frac{1}{x}\right)^5\left(x-\frac{1}{x}\right)^5\left(x-\frac{1}{x}\right)^2$$

$$= \left[\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)\right]^5\left[x-\frac{1}{x}\right]^2$$

$$= \left[x^2-\frac{1}{x^2}\right]^5\left[x-\frac{1}{x}\right]^2$$

$$= \left[(x^2)^5 - 5(x^2)^4 \cdot \frac{1}{x^2} + \frac{5\cdot 4}{2\cdot 1}(x^2)^3 \cdot \frac{1}{(x^2)^2} - \frac{5\cdot 4\cdot 3}{3\cdot 2\cdot 1}(x^2)^2 \cdot \frac{1}{(x^2)^3} + \frac{5\cdot 4\cdot 3\cdot 2}{4\cdot 3\cdot 2}\, x^2 \cdot \frac{1}{(x^2)^4} - \frac{1}{(x^2)^5}\right]\left[x^2 + \frac{1}{x^2} - 2\right]$$

$$-2^{12} i \cos^5\theta \sin^7\theta = \left[x^{10} - 5x^6 + 10x^2 - \frac{10}{x^2} + \frac{5}{x^6} - \frac{1}{x^{10}}\right]\left(x^2 + \frac{1}{x^2} - 2\right)$$

$$-2^{12} i \cos^5\theta \sin^7\theta = x^{12} - 5x^8 + 10x^4 - 10 + \frac{5}{x^4} - \frac{1}{x^8} + x^8 - 5x^4 + 10 - \frac{10}{x^4} + \frac{5}{x^8} - \frac{1}{x^{12}} - 2x^{10} + 10x^6 - 20x^2 + \frac{20}{x^2} - \frac{10}{x^6} + \frac{2}{x^{10}}$$

1.2-25

$$= \left(x^{12} - \frac{1}{x^{12}}\right) - 2\left(x^{10} - \frac{1}{x^{10}}\right) - 4\left(x^8 - \frac{1}{x^8}\right) + 10\left(x^6 - \frac{1}{x^6}\right) + 5\left(x^4 - \frac{1}{x^4}\right) - 20\left(x^2 - \frac{1}{x^2}\right)$$

$$= 2i \sin 12\theta - 2(2i \sin 10\theta) - 4(2i \sin 8\theta) + 10(2i \sin 6\theta) + 5(2i \sin 4\theta) - 20(2i \sin 2\theta)$$

$$-2^{12} i \cos^5\theta \sin^7\theta = 2i\left(\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta\right)$$

$$\Rightarrow \boxed{\cos^5\theta \sin^7\theta = -\frac{1}{2^{11}}\left(\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta\right)}$$

—x— Ans —x—

**Q.11** Show that $\cos^4\theta + \sin^4\theta = \frac{1}{4}(\cos 4\theta + 3)$

**Sol:-** let $x = \cos\theta + i\sin\theta$ then $\frac{1}{x} = \cos\theta - i\sin\theta$ $\Rightarrow$ $2\cos\theta = x + \frac{1}{x}$, $2i\sin\theta = x - \frac{1}{x}$

So $(2\cos\theta)^4 = \left(x + \frac{1}{x}\right)^4$

$$2^4\cos^4\theta = x^4 + \cancel{4x^3\cdot\frac{1}{x}} + \frac{4\cdot 3}{2\cdot 1}x^2\cdot\frac{1}{x^2} + \cancel{\frac{4\cdot 3\cdot 2}{3\cdot 2\cdot 1}x^1\cdot\frac{1}{x^3}} + \frac{4\cdot 3\cdot 2\cdot 1}{4\cdot 3\cdot 2\cdot 1}\cdot\frac{1}{x^4} \rightarrow (1)$$

$$(2i\sin\theta)^4 = \left(x - \frac{1}{x}\right)^4 = x^4 - \cancel{4x^3\cdot\frac{1}{x}} + \frac{4\cdot 3}{2\cdot 1}x^2\cdot\frac{1}{x^2} - \cancel{\frac{4\cdot 3\cdot 2}{3\cdot 2\cdot 1}x\cdot\frac{1}{x^3}} + \frac{1}{x^4} \rightarrow (2)$$

+ (1) and (2), we get

$$2^4\cos^4\theta + 2^4 i^4 \sin^4\theta = 2\left\{x^4 + \frac{4\cdot 3}{2\cdot 1} + \frac{1}{x^4}\right\}$$

$$2^4\left[\cos^4\theta+\sin^4\theta\right]=2\left\{x^4+6+\frac{1}{x^4}\right\} \qquad (2^4=1)$$

$$\cos^4\theta+\sin^4\theta=\frac{1}{2^3}\left[\left(x^4+\frac{1}{x^4}\right)+6\right]=\frac{1}{8}\left[2\cos4\theta+6\right]$$

$$\boxed{\cos^4\theta+\sin^4\theta=\frac{1}{4}\left[\cos4\theta+3\right]}$$

x —— Ans —— x

$\boxed{1.2-26}$

**Q.12** Prove that $64(\cos^8\theta+\sin^8\theta)=\cos8\theta+28\cos4\theta+35$

**Sol:** Let $x=\cos\theta+i\sin\theta$, $\frac{1}{x}=\cos\theta-i\sin\theta$ $\Rightarrow$ $x+\frac{1}{x}=2\cos\theta$ and $x-\frac{1}{x}=2i\sin\theta$

So.

$$(2\cos\theta)^8+(2i\sin\theta)^8=\left[\left(x+\frac{1}{x}\right)^8+\left(x-\frac{1}{x}\right)^8\right]$$

$$2^8\left[\cos^8\theta+\sin^8\theta\right]=\begin{cases} x^8+8\cancel{x^7}\cdot\frac{1}{x}+\frac{8\cdot7}{2}x^6\cdot\frac{1}{x^2}+\frac{8\cdot7\cdot6}{3\cdot2}\cancel{x^5}\cdot\frac{1}{x^3}+\frac{8\cdot7\cdot6\cdot5}{4\cdot3\cdot2}x^4\cdot\frac{1}{x^4} \\ +\frac{8\cdot7\cdot6\cdot5\cdot4}{5\cdot4\cdot3\cdot2}x^3\cdot\frac{1}{x^5}+\frac{8\cdot7\cdot6\cdot5\cdot4\cdot3}{6\cdot5\cdot4\cdot3\cdot2}x^2\cdot\frac{1}{x^6}+\frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2}{7\cdot6\cdot5\cdot4\cdot3\cdot2}x\cdot\frac{1}{x^7}+\frac{1}{x^8} \\ x^8-8\cancel{x^7}\frac{1}{x}+\frac{8\cdot7}{2}x^6\cdot\frac{1}{x^2}-\frac{8\cdot7}{}x^5\cdot\frac{1}{x^3}+\frac{8\cdot7\cdot6\cdot5}{4\cdot3\cdot2}x^4\cdot\frac{1}{x^4} \\ -\frac{8\cdot7\cdot6\cdot5\cdot4}{5\cdot4\cdot3\cdot2}x^3\cdot\frac{1}{x^5}+\frac{8\cdot7\cdot6\cdot5\cdot4\cdot3}{6\cdot5\cdot4\cdot3\cdot2}x^2\cdot\frac{1}{x^6}-\frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2}{7\cdot6\cdot5\cdot4\cdot3\cdot2}x\frac{1}{x^7}+\frac{1}{x^8} \end{cases}$$

$$2^8\left[\cos^8\theta+\sin^8\theta\right]=2\left[x^8+28\,x^6\cdot\frac{1}{x^2}+70+28\cdot x^2\cdot\frac{1}{x^6}+\frac{1}{x^8}\right]$$

$$=2\left[\left(x^8+\frac{1}{x^8}\right)+28\left(x^4+\frac{1}{x^4}\right)+70\right]$$

$$=2\left[2\cos8\theta+28\left(2\cos4\theta\right)+70\right]$$

$$2^6(\cos^8\theta+\sin^8\theta)=\cancel{2}\left[\cos8\theta+28\cos4\theta+35\right]$$

$$\boxed{\Rightarrow\ 64(\cos^8\theta+\sin^8\theta)=\cos8\theta+28\cos4\theta+35}$$

Proved.

**Q-13** PROVE THAT:- [1-2-27]

**P-(i)** $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

**PROOF** Let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$

$\Rightarrow x - \frac{1}{x} = 2i\sin\theta$

Thus $(2i\sin\theta)^3 = \left(x - \frac{1}{x}\right)^3 = x^3 - 3x + \frac{3}{x} - \frac{1}{x^3}$

$2^3 i^3 \sin^3\theta = \left(x^3 - \frac{1}{x^3}\right) - 3\left(x - \frac{1}{x}\right)$

$-2^3 i \sin^3\theta = 2i\sin 3\theta - 3(2i\sin\theta)$

$-8i\sin^3\theta = 2i\sin 3\theta - 6i\sin\theta$

$\Rightarrow -4\sin^3\theta = \sin 3\theta - 3\sin\theta$

$\Rightarrow \boxed{\sin 3\theta = 3\sin\theta - 4\sin^3\theta}$

Proved

**Part-(ii)** $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

**PROOF:-** Let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$

$\Rightarrow x + \frac{1}{x} = 2\cos\theta$

Thus $(2\cos\theta)^3 = \left(x + \frac{1}{x}\right)^3 = x^3 + 3x^2 \cdot \frac{1}{x} + 3x \cdot \frac{1}{x^2} + \frac{1}{x^3}$

$2^3\cos^3\theta = x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}$

$2^3\cos^3\theta = \left(x^3 + \frac{1}{x^3}\right) + 3\left(x + \frac{1}{x}\right)$

$\cancel{2^3}\,2\cos^3\theta = 2\cos 3\theta + 3(2\cos\theta)$

$2^2\cos^3\theta = \cos 3\theta + 3\cos\theta$

$\Rightarrow \boxed{\cos 3\theta = 4\cos^3\theta - 3\cos\theta}$

—x— Proved —x—

## ALTERNATE OF PART-(i) AND PART-(ii)

$\because (\cos 3\theta + i\sin 3\theta) = (\cos\theta + i\sin\theta)^3$

(By using De-moivre's Th.)

But $(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3 \sin^3\theta$

$$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$

or $\cos 3\theta + i\sin 3\theta = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

$$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

$$= [\cos^3\theta - 3\cos\theta(1-\cos^2\theta)] + i[3\sin\theta(1-\sin^2\theta) - \sin^3\theta]$$

$$= [\cos^3\theta - 3\cos\theta + 3\cos^3\theta] + i[3\sin\theta - 3\sin^3\theta - \sin^3\theta]$$

$$\cos 3\theta + i\sin 3\theta = [4\cos^3\theta - 3\cos\theta] + i[3\sin\theta - 4\sin^3\theta]$$

Equating real and imaginary parts we get

$$\boxed{\begin{aligned}\cos 3\theta &= 4\cos^3\theta - 3\cos\theta \\ \text{and} \quad \sin 3\theta &= 3\sin\theta - 4\sin^3\theta\end{aligned}}$$

Proved

$\boxed{1\cdot 2\text{-}28}$

PART-(iii)&(iv) $\sin 4\theta = 4(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta)$

and $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

PROOF:-

$\because (\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta \longrightarrow (i)$

(De-Moivre's Th.)

but $(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4i\cos^3\theta\sin\theta + \frac{4\cdot 3}{2}i^2\cos^2\theta\sin^2\theta$

$+ \frac{4\cdot3\cdot2}{3\cdot2}i^3\cos\theta\sin^3\theta + i^4\sin^4\theta$

$$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$$

(Binomial Th.)

or $(\cos\theta + i\sin\theta)^4 = (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$

using (i)

$\cos 4\theta + i\sin 4\theta = [\cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + \sin^4\theta] + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$

$\cos 4\theta + i\sin 4\theta = [\cos^4\theta + 6\cos^4\theta - 6\cos^2\theta + (1-\cos^2\theta)^2] + i[4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta]$

$$= [7\cos^4\theta - 6\cos^2\theta + 1 + \cos^4\theta - 2\cos^2\theta] + i[4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta]$$

$\cos 4\theta + i\sin 4\theta = [8\cos^4\theta - 8\cos^2\theta + 1] + i[4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta]$

Equating real and imaginary parts, we get

$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$ ⟶ Part (iv)

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$ ⟶ Part (iii)

$\boxed{12-29}$

(v) $\frac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1$

1st Method

$x^5 = \cos 5\theta + i\sin 5\theta =$
$\frac{1}{x^5} = \cos 5\theta - i\sin 5\theta$
$x^5 - \frac{1}{x^5} = 2i\sin 5\theta$

$\left(x - \frac{1}{x}\right)^5 = 2^5 i^5 \sin^5\theta$

$32 i\sin^5\theta = x^5 - 5x^4\frac{1}{x} + \frac{5\cdot4}{2}x^3\frac{1}{x^2} - \frac{5\cdot4\cdot3}{1\cdot2\cdot3}x^2\frac{1}{x^3} + \frac{5\cdot4\cdot3\cdot2}{1\cdot2\cdot3\cdot4}x\frac{1}{x^4} - \frac{1}{x^5}$

$= \left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)$

$= (2i\sin 5\theta) - 5(2i\sin 3\theta) + 10(2i\sin\theta)$

$32 i\sin^5\theta = 2i(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$

$\sin\theta(16\sin^4\theta) + 5(3\sin\theta - 4\sin^3\theta) - 10\sin\theta = \sin 5\theta$

$\frac{\sin\theta}{\sin\theta}(16\sin^4\theta + 15 - 20\sin^2\theta - 10) = \frac{\sin 5\theta}{\sin\theta}$

$16(1-\cos^2\theta)^2 + 15 - 20(1-\cos^2\theta) - 10$

$16(1 + \cos^4\theta - 2\cos^2\theta) + 15 - 20 + 20\cos^2\theta - 10$

$16 + 16\cos^4\theta - 32\cos^2\theta - 20 + 20\cos^2\theta - 10$

$1 + 16\cos^4\theta - 12\cos^2\theta = \frac{\sin 5\theta}{\sin\theta}$

proved

2nd Method.

**Part-(V)** $\dfrac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1$

$\boxed{1\cdot 2\text{-}30}$

**PROOF** According to De moivre's Th.

$$[\cos\theta + i\sin\theta]^5 = \cos 5\theta + i\sin 5\theta \longrightarrow (1)$$

but $[\cos\theta + i\sin\theta]^5 = \cos^5\theta + 5i\cos^4\theta\cdot\sin\theta + \frac{5\cdot 4}{2\cdot 1}\cos^3\theta\sin^2\theta - \frac{5\cdot 4\cdot 3}{3\cdot 2\cdot 1}i\cos^2\theta\sin^3\theta$

by Binomial Th

$$+\frac{5\cdot 4\cdot 3\cdot 2}{4\cdot 3\cdot 2\cdot 1}\cos\theta\sin^4\theta + i\sin^5\theta$$

$$[\cos\theta + i\sin\theta]^5 = \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$$

using (1), we get

$$\cos 5\theta + i\sin 5\theta = (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$$

Equating imaginary parts, we get

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

$$\Rightarrow \frac{\sin 5\theta}{\sin\theta} = \frac{\sin\theta[5\cos^4\theta - 10\cos^2\theta\sin^2\theta + \sin^4\theta]}{\sin\theta}$$

by $\sin\theta$

$$= 5\cos^4\theta - 10\cos^2\theta\sin^2\theta + \sin^4\theta$$

$$= 5\cos^4\theta - 10\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$$

$$= 5\cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + \cos^4\theta + 1 - 2\cos^2\theta$$

or $$\frac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1$$

x———— Proved ————x

**Q.14** Prove that $\tan 6\theta = 2t\left(\dfrac{3-10t^2+3t^4}{1-15t^2+15t^4-t^6}\right)$ where $t=\tan\theta$

**PROOF** According to Demoivre's Th.

$$(\cos\theta + i\sin\theta)^6 = \cos 6\theta + i\sin 6\theta \longrightarrow (1)$$

but $(\cos\theta + i\sin\theta)^6 = \Big\{\cos^6\theta + 6i\cos^5\theta\sin\theta - \frac{6\cdot 5}{2}\cos^4\theta\sin^2\theta - \frac{6\cdot 5\cdot 4}{3\cdot 2}i\cos^3\theta\sin^3\theta$

by Binomial Th

$$+\frac{6\cdot 5\cdot 4\cdot 3}{4\cdot 3\cdot 2}\cos^2\theta\sin^4\theta + \frac{6\cdot 5\cdot 4\cdot 3\cdot 2}{5\cdot 4\cdot 3\cdot 2}i\cos\theta\sin^5\theta - \sin^6\theta\Big\}$$

$$\cos 6\theta + i\sin 6\theta = \left\{\begin{array}{l}\cos^6\theta + 6i\cos^5\theta\sin\theta - 15\cos^4\theta\sin^2\theta - 20i\cos^3\theta\sin^3\theta \\ + 15\cos^2\theta\sin^4\theta + 6i\cos\theta\sin^5\theta - \sin^6\theta\end{array}\right\}$$

$\boxed{1.2\text{-}31}$

Equating real and imaginary parts.

$$\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta \longrightarrow \text{(ii)}$$

and $$\sin 6\theta = 6\cos^5\theta\sin\theta - 20\sin^3\theta\cos^3\theta + 6\cos\theta\sin^5\theta$$

$$\frac{\sin 6\theta}{\cos 6\theta} = \left(\frac{6\cos^5\theta\sin\theta - 20\sin^3\theta\cos^3\theta + 6\cos\theta\sin^5\theta}{\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta}\right)$$

$$\tan 6\theta = \frac{\dfrac{6\cos^5\theta\sin\theta - 20\sin^3\theta\cos^3\theta + 6\cos\theta\sin^5\theta}{\cos^6\theta}}{\dfrac{\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta}{\cos^6\theta}}$$

$$= \frac{6\tan\theta - 20\tan^3\theta + 6\tan^5\theta}{1 - 15\tan^2\theta + 15\tan^4\theta - \tan^6\theta}$$

R.H.S.

$$= \frac{2\tan\theta(3 - 10\tan^2\theta + 3\tan^4\theta)}{1 - 15\tan^2\theta + 15\tan^4\theta - \tan^6\theta}$$

$$= \frac{2t(3 - 10t^2 + 3t^4)}{1 - 15t^2 + 15t^4 - t^6}$$

**Q.15** Prove that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$ and hence solve the equation $1-3t^2 = 3t - t^3$

**Sol:** Since $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta \rightarrow (1)$

but $(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$ by B.Th.

using (1), we get

$\cos 3\theta + i\sin 3\theta = (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$

$= [\cos^3\theta - 3\cos\theta(1-\cos^2\theta)] + i[3(1-\sin^2\theta)\sin\theta - \sin^3\theta]$

$= [\cos^3\theta - 3\cos\theta + 3\cos^3\theta] + i[3\sin\theta - 3\sin^3\theta - \sin^3\theta]$

$\cos 3\theta + i\sin 3\theta = [4\cos^3\theta - 3\cos\theta] + i[3\sin\theta - 4\sin^3\theta]$

Equating real and imaginary parts, we get

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta, \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\boxed{1.2-32}$

$\Rightarrow \dfrac{\sin 3\theta}{\cos 3\theta} = \dfrac{3\sin\theta - 4\sin^3\theta}{4\cos^3\theta - 3\cos\theta}$

$= \dfrac{\sin\theta(3-4\sin^2\theta)}{\cos\theta(4\cos^2\theta - 3)} = \dfrac{\tan\theta(3-4\sin^2\theta)}{4\cos^2\theta - 3}$

$= \dfrac{\tan\theta[3\cdot 1 - 4\sin^2\theta]}{4\cos^2\theta - 3\cdot 1} = \dfrac{\tan\theta[3(\cos^2\theta + \sin^2\theta) - 4\sin^2\theta]}{4\cos^2\theta - 3(\sin^2\theta + \cos^2\theta)}$

$= \dfrac{\tan\theta[3\cos^2\theta - \sin^2\theta]}{\cos^2\theta - 3\sin^2\theta}$

$\div$ N+D by $\cos^2\theta$

$\tan 3\theta = \dfrac{\tan\theta(3-\tan^2\theta)}{1-3\tan^2\theta}$ proved

Now Put $\tan\theta = t$

$\Rightarrow \tan 3\theta = \dfrac{t(3-t^2)}{1-3t^2} = \dfrac{3t - t^3}{1-3t^2} \rightarrow (2)$

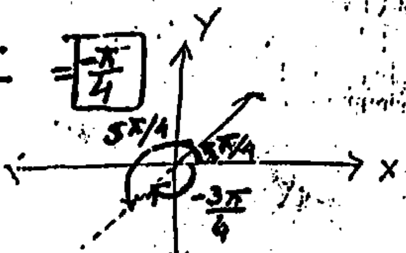
Since we are asked to solve $1-3t^2 = 3t - t^3$

$\Rightarrow 1 = \dfrac{3t - t^3}{1-3t^2} \rightarrow (3) \Rightarrow \tan 3\theta = 1$ (from (2) and (3))

$3\theta = \tan^{-1}(1) = \pi/4, \frac{5\pi}{4}, -\frac{3\pi}{4}$

$\Rightarrow 3\theta = \frac{\pi}{4}, \quad 3\theta = \frac{5\pi}{4}, \quad$ and $\quad 3\theta = -\frac{3\pi}{4}$

$\Rightarrow \boxed{\theta = \frac{\pi}{12}}, \quad \boxed{\theta = \frac{5\pi}{12}}, \quad \theta = \frac{-3\pi}{(3)4} = \boxed{\frac{-\pi}{4}}$




Since $t = \tan\theta$

So $t = \tan\frac{\pi}{12}$, $\quad t = \tan\frac{5\pi}{12}$ & $\boxed{t = \tan(\frac{-\pi}{4}) = -1}$

$\because \tan\frac{\pi}{6} = \tan\left(\frac{\pi}{12} + \frac{\pi}{12}\right)$ $\quad \left(\because \tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}\right)$

$\Rightarrow \frac{1}{\sqrt{3}} = \frac{2\tan\frac{\pi}{12}}{1-\tan^2\frac{\pi}{12}}$ $\quad \boxed{\because \sin\frac{\pi}{6} = \frac{1}{2} \;\&\; \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}}$

$\Rightarrow 1 - \tan^2\frac{\pi}{12} = 2\sqrt{3}\tan\frac{\pi}{12}$

or $\tan^2\frac{\pi}{12} + 2\sqrt{3}\tan\frac{\pi}{12} = 1$

$\Rightarrow \tan^2\pi/12 + 2\sqrt{3}\tan\pi/12 + 3 = 1+3$ (completing square)

$\Rightarrow \left(\tan\frac{\pi}{12} + \sqrt{3}\right)^2 = 2^2$

$\Rightarrow \tan\frac{\pi}{12} + \sqrt{3} = 2 \quad \Rightarrow \quad \boxed{\tan\frac{\pi}{12} = 2 - \sqrt{3} = t}$

Also $\tan\frac{5\pi}{6} = \tan\left(\frac{5\pi}{12} + \frac{5\pi}{12}\right)$ $\quad \boxed{\begin{array}{l}\sin\frac{5\pi}{6} = \frac{1}{2} \\ \Rightarrow \cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} \\ \Rightarrow \tan\frac{5\pi}{6} = -\frac{1}{\sqrt{3}}\end{array}}$

or $\frac{-1}{\sqrt{3}} = \frac{2\tan\frac{5\pi}{12}}{1-\tan^2\frac{5\pi}{12}}$

$-1 + \tan^2\frac{5\pi}{12} = 2\sqrt{3}\tan\frac{5\pi}{12}$

or $\tan^2\frac{5\pi}{12} - 2\sqrt{3}\tan\frac{5\pi}{12} = 1$

Completing sq. we get

$\tan^2\frac{5\pi}{12} - 2\sqrt{3}\tan\frac{5\pi}{12} + (\sqrt{3})^2 = 1+3$

$\left(\tan\frac{5\pi}{12} - \sqrt{3}\right)^2 = 2^2$

$\Rightarrow \tan\frac{5\pi}{12} - \sqrt{3} = 2$

$\Rightarrow \boxed{t = \tan\frac{5\pi}{12} = 2 + \sqrt{3}}$

Hence the required roots of cubic equation $1 - 3t^2 = 3t - t^3$ are

$-1, \; 2+\sqrt{3}, \; 2-\sqrt{3}$

ans

## Q.16 Prove that $\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}$ [1.2-34]

**Sol** Consider the seventh roots of unity

i.e let $x^7 = 1 \Rightarrow x^7 = 1 + 0i$

$\Rightarrow x^7 = \cos 0 + i\sin 0 = \cos(0 + 2k\pi) + i\sin(0 + 2\pi k)$

or $x^7 = \cos 2\pi k + i\sin 2k\pi$

So seven 7th roots of unity are

$$x_k = (\cos 2\pi k + i\sin 2k\pi)^{\frac{1}{7}}$$

where $k = 0, \pm 1, \pm 2, \pm 3$

$\Rightarrow x_k = \cos\frac{2\pi k}{7} + i\sin\frac{2k\pi}{7} \to (1)$, where $k = 0, \pm1, \pm2, \pm3$

after putting values of $k$ in (1), we get its seven roots

i.e $1,\ \cos\frac{2\pi}{7} \pm i\sin\frac{2\pi}{7},\ \cos\frac{4\pi}{7} \pm i\sin\frac{4\pi}{7},\ \cos\frac{6\pi}{7} \pm i\sin\frac{6\pi}{7}$

(Since $\sin(-\alpha) = -\sin\alpha$ and $\cos(-\alpha) = \cos\alpha$)

Now from theory of equations, the sum of root of $x^7 - 1 = 0$ is zero

$$\Rightarrow 1 + \left(\cos\frac{2\pi}{7} + i\cancel{\sin\frac{2\pi}{7}}\right) + \left(\cos\frac{4\pi}{7} + i\cancel{\sin\frac{4\pi}{7}}\right) + \left(\cos\frac{6\pi}{7} + i\cancel{\sin\frac{6\pi}{7}}\right.$$
$$\left.\cos\frac{2\pi}{7} - i\cancel{\sin\frac{2\pi}{7}} + \cos\frac{4\pi}{7} - i\cancel{\sin\frac{4\pi}{7}} + \cos\frac{6\pi}{7} - i\cancel{\sin\frac{6\pi}{7}}\right) = 0$$

$\Rightarrow 1 + 2\cos\frac{2\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{6\pi}{7} = 0$

$\Rightarrow 2\left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}\right) = -1$

$\boxed{\cos(\pi - \alpha) = -\cos\alpha}$

$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$

$\cos\frac{2\pi}{7} + \cos\left(\pi - \frac{3\pi}{7}\right) + \cos\left(\pi - \frac{\pi}{7}\right) = -\frac{1}{2}$

$\pi - \frac{3\pi}{7} = \frac{7\pi - 3\pi}{7} = \frac{4\pi}{7}$

$\pi - \frac{\pi}{7} = \frac{7\pi - \pi}{7} = \frac{6\pi}{7}$

$\cos\frac{2\pi}{7} - \cos\frac{3\pi}{7} - \cos\frac{\pi}{7} = -\frac{1}{2}$

$\Rightarrow \boxed{\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}}$ Proved

**Q.17** Prove the following relations ($\forall$ $m, n$ belongs to $\mathbb{Z}$ mean $m$ & $n$ are integers)

**(i)** $z^m z^n = z^{m+n}$ $\boxed{1\cdot2\text{-}35}$

**PROOF** Let $z = r(\cos\theta + i\sin\theta)$

$\Rightarrow z^m = r^m(\cos\theta + i\sin\theta)^m$

$z^m = r^m(\cos m\theta + i\sin m\theta)$

Similarly $z^n = r^n(\cos n\theta + i\sin n\theta)$

Then L.H.S. $= z^m \cdot z^n$

$= r^m[\cos m\theta + i\sin m\theta]\, r^n[\cos n\theta + i\sin n\theta]$

$= r^m r^n[\cos m\theta + i\sin m\theta][\cos n\theta + i\sin n\theta]$

$= r^{m+n}\begin{bmatrix}(\cos m\theta\cos n\theta - \sin m\theta\sin n\theta) \\ + i(\sin m\theta\cos n\theta + \sin n\theta\cos m\theta)\end{bmatrix}$

$= r^{m+n}[\cos(m\theta + n\theta) + i\sin(m\theta + n\theta)]$

$= r^{m+n}[\cos(m+n)\theta + i\sin(m+n)\theta]$

$= r^{m+n}(\cos\theta + i\sin\theta)^{m+n}$ by Demoivre's Th.

$= z^{m+n} =$ R.H.S.

x———————x

**(ii)** $(z^m)^n = z^{mn}$

**PROOF:-** Let $z = r(\cos\theta + i\sin\theta)$

$\Rightarrow z^m = r^m(\cos\theta + i\sin\theta)^m$ (De-moivre's Th.)

$z^m = r^m(\cos m\theta + i\sin m\theta)$

$\Rightarrow (z^m)^n = r^{mn}(\cos m\theta + i\sin m\theta)^n$ (De-moivres Th.)

$$L.H.S = \left(Z^m\right)^n = r^{mn}\left[\cos mn\alpha + i \sin mn\alpha\right] = r^{mn}\left[\cos\alpha + i\sin\alpha\right]^{mn}$$

$$= Z^{mn} = R.H.S.$$

---

<u>**(iii)**</u> $(Z_1 Z_2)^n = Z_1^n Z_2^n$

<u>PROOF</u> let $Z_1 = r_1[\cos\theta_1 + i\sin\theta_1]$ and $Z_2 = r_2[\cos\theta_2 + i\sin\theta_2]$

Hence $Z_1 \cdot Z_2 = r_1[\cos\theta_1 + i\sin\theta_1] \cdot r_2[\cos\theta_2 + i\sin\theta_2]$

$$= r_1 r_2 [\cos\theta_1 + i\sin\theta_1][\cos\theta_2 + i\sin\theta_2]$$

$$= r_1 r_2 [(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$

$$Z_1 Z_2 = r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

$$L.H.S = (Z_1 Z_2)^n = r_1^n r_2^n [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]^n$$

by Demoivre th.

$$\boxed{\begin{aligned} &\because Z = r(\cos\alpha + i\sin\alpha) \\ &\Rightarrow Z^n = [r(\cos\alpha + i\sin\alpha)]^n \\ &= r^n(\cos\alpha + i\sin\alpha)^n \\ &= r^n[\cos n\alpha + i\sin n\alpha] \end{aligned}}$$

$$= r_1^n r_2^n [\cos n(\theta_1+\theta_2) + i\sin n(\theta_1+\theta_2)]$$

$$= r_1^n r_2^n [\cos(n\theta_1 + n\theta_2) + i\sin(n\theta_1 + n\theta_2)]$$

$$= r_1^n r_2^n [(\cos n\theta_1\cos n\theta_2 - \sin n\theta_1\sin n\theta_2) + i(\sin n\theta_1\cos n\theta_2 + \cos n\theta_1\sin n\theta_2)]$$

$$= r_1^n r_2^n [(\cos n\theta_1\cos n\theta_2 + i^2\sin n\theta_1\sin n\theta_2) + (i\sin n\theta_1\cos n\theta_2 + i\cos n\theta_1\sin n\theta_2)]$$

$$= r_1^n r_2^n [(\cos n\theta_1\cos n\theta_2 + i\cos n\theta_1\sin n\theta_2) + (i\sin n\theta_1\cos n\theta_2 + i^2\sin n\theta_1\sin n\theta_2)]$$

$$= r_1^n r_2^n [\cos n\theta_1(\cos n\theta_2 + i\sin n\theta_2) + i\sin n\theta_1(\cos n\theta_2 + i\sin n\theta_2)]$$

$$= r_1^n r_2^n [(\cos n\theta_1 + i\sin n\theta_1)(\cos n\theta_2 + i\sin n\theta_2)]$$

$$= r_1^n(\cos n\theta_1 + i\sin n\theta_1) \cdot r_2^n(\cos n\theta_2 + i\sin n\theta_2)$$

$$= r_1^n(\cos\theta_1 + i\sin\theta_1)^n \cdot r_2^n(\cos\theta_2 + i\sin\theta_2)^n$$

$$= Z_1^n Z_2^n = R.H.S$$

(iv) $\frac{z^m}{z^n} = z^{m-n}$, $z \neq 0$

<u>Proof</u> let $z = r(\cos\theta + i\sin\theta)$

$\Rightarrow z^m = r^m(\cos\theta + i\sin\theta)^m$ & $z^n = r^n(\cos\theta + i\sin\theta)^n$

$$L.H.S = \frac{z^m}{z^n} = \frac{r^m(\cos\theta + i\sin\theta)^m}{r^n(\cos\theta + i\sin\theta)^n}$$

$$= r^{m-n} \cdot \frac{(\cos m\theta + i\sin m\theta)}{(\cos n\theta + i\sin n\theta)}$$

$$= r^{m-n} \cdot (\cos m\theta + i\sin m\theta)(\cos n\theta + i\sin n\theta)^{-1}$$

$$= r^{m-n}[\cos m\theta + i\sin m\theta][\cos(-n\theta) + i\sin(-n\theta)]$$

$$= r^{m-n}[\cos m\theta + i\sin m\theta][\cos n\theta - i\sin n\theta].$$

$$= r^{m-n}\big[(\cos m\theta\cos n\theta + \sin m\theta\sin n\theta) + i(\sin m\theta\cos n\theta - \cos m\theta\sin n\theta)\big]$$

$$= r^{m-n}[\cos(m\theta - n\theta) + i\sin(m\theta - n\theta)]$$

$$= r^{m-n}[\cos(m-n)\theta + i\sin(m-n)\theta]$$

$$= r^{m-n}(\cos\theta + i\sin\theta)^{m-n}$$ using Demoivre Th.

$$= z^{m-n} = R.H.S.$$

(v) $\left(\frac{z_1}{z_2}\right)^n = \frac{z_1^n}{z_2^n}$, $z_2 \neq 0$

<u>Proof</u> let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$\Rightarrow \frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

1.2-38

$$L.H.S = \left(\frac{z_1}{z_2}\right)^n = \left[\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}\right]^n$$

$$\Rightarrow \left(\frac{z_1}{z_2}\right)^n = \frac{r_1^n}{r_2^n}\left[\frac{(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}\right]^n$$

$$= \frac{r_1^n}{r_2^n}\left[\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1)}{\cos^2\theta_2 + \sin^2\theta_2}\right]^n$$

$$= \frac{r_1^n}{r_2^n}\left[\frac{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)}{1}\right]^n$$

$$= \frac{r_1^n}{r_2^n}\left[\cos n(\theta_1 - \theta_2) + i\sin n(\theta_1 - \theta_2)\right]$$

$$= \frac{r_1^n}{r_2^n}\left[\cos(n\theta_1 - n\theta_2) + i\sin(n\theta_1 - n\theta_2)\right]$$

$$= \frac{r_1^n}{r_2^n}\left[(\cos n\theta_1\cos n\theta_2 + \sin n\theta_1\sin n\theta_2) + i(\sin n\theta_1\cos n\theta_2 - \sin n\theta_2\cos n\theta_1)\right]$$

$$= \frac{r_1^n}{r_2^n}\left[(\cos n\theta_1\cos n\theta_2 - i^2\sin n\theta_1\sin n\theta_2) + i(\sin n\theta_1\cos n\theta_2 - \sin n\theta_2\cos n\theta_1)\right]$$

$$= \frac{r_1^n}{r_2^n}\left[(\cos n\theta_1\cos n\theta_2 - i\sin n\theta_2\cos n\theta_1) + (i\sin n\theta_1\cos n\theta_2 - i^2\sin n\theta_1\sin n\theta_2)\right]$$

$$= \frac{r_1^n}{r_2^n}\left[\cos n\theta_1(\cos n\theta_2 - i\sin n\theta_2) + i\sin n\theta_1(\cos n\theta_2 - i\sin n\theta_2)\right]$$

$$= \frac{r_1^n}{r_2^n}\left[(\cos n\theta_1 + i\sin n\theta_1)(\cos\theta_2 n - i\sin n\theta_2)\right]$$

$$= \frac{r_1^n}{r_2^n} \times (\cos\theta_1 + i\sin\theta_1)^n(\cos\theta_2 n + i\sin n\theta_2)^{-1}$$

$$= \frac{r_1^n}{r_2^n}\frac{(\cos\theta_1 + i\sin\theta_1)^n}{(\cos n\theta_2 + i\sin n\theta_2)} = \frac{r_1^n}{r_2^n} \times \frac{(\cos\theta_1 + i\sin\theta_1)^n}{(\cos\theta_2 + i\sin\theta_2)^n} = \frac{z_1^n}{z_2^n}$$

$$= R.H.S$$

# The Complex Number System.

The set $C = R \times R = \{(a,b) / a,b \in R\}$ is called the set of complex number if the following conditions are satisfied.

i) $(a,b)+(c,d) = (a+c,\ b+d)$ (Addition)

ii) $(a,b)\cdot(c,d) = (ac-bd,\ ad+bc)$ (Multiplication)

iii) $K(a,b) = (Ka,\ Kb)$ where $K \in R$ (Scalar Multiple)

iv) $(a,b) = (c,d)$ iff $a=c,\ b=d$ (Equality)

**Note.** $\boxed{(a,b) = a+bi}$ $a$ = Real Part, $b$ = Imag Part

$i = (0,1)$

$(a,b) = (a,0)+(0,b)$
$= a(1,0)+b(0,1)$
$= a(1)+b(i)$
$(a,b) = a+bi$

$i = (0,1)$
$i \cdot i = (0,1)(0,1)$
$i^2 = (0\cdot0-1\cdot1,\ 0\cdot1+1\cdot0)$
$i^2 = (-1,0)$
$i^2 = -1+0i$
$i^2 = -1$

where $(a,b) \in C$ & $a+bi \in C$
$\because (a,b) = a(1,0)+b(0,1)$
$(a,b) = a\cdot1 + b\,i = a+bi$

**Multiplicative Identity in C.** $(1,0) = 1 = 1+0i$

**Additive Identity in C.** $(0,0) = 0 = 0+0i$

**Additive Inverse of $(a,b)$** is $(-a,-b)$

**Multiplicative Inverse of $(a,b) \in C$**

Let $z = (a,b) = a+ib$

$z^{-1} = (a+ib)^{-1} = \dfrac{1}{a+ib}$

$= \dfrac{1}{a+ib} \times \dfrac{(a-ib)}{(a-ib)}$

$z^{-1} = \dfrac{a-ib}{a^2+b^2} = \dfrac{a}{a^2+b^2} - \dfrac{ib}{a^2+b^2}$

$z^{-1} = \left(\dfrac{a}{a^2+b^2},\ \dfrac{-b}{a^2+b^2}\right)$

(To verify $zz^{-1} = (1,0) = 1$)

**Modulus of $(a,b) \in C$**

If $z = (a,b)$ then $|z| = \sqrt{a^2+b^2}$

**Conjugate of $(a,b) \in C$**

If $z = (a,b) = a+bi$
then $\bar{z} = \overline{a+bi} = a-bi$

**To Prove** $z\cdot\bar{z} = |z|^2$ $= (\text{Re}\,z)^2+(\text{Im}\,z)^2$

LHS $z\cdot\bar{z}$
$= (a+bi)(a-bi)$
$= a^2+b^2 = (\text{Re}\,z)^2+(\text{Im}\,z)^2$

$\boxed{z\bar{z} = |z|^2}$

**Also** $z^2 = (a+bi)(a+bi)$
$z^2 = a^2-b^2+2abi$
$|z^2| = \sqrt{(a^2-b^2)^2+(2ab)^2}$
$= \sqrt{a^4+b^4-2a^2b^2+4a^2b^2}$
$= \sqrt{a^4+b^4+2a^2b^2}$
$= \sqrt{(a^2+b^2)^2} = a^2+b^2 = |z|^2$

$\therefore \boxed{|z^2| = |z|^2}$

Imp

**Th** Let $z_1, z_2$ be complex numbers. Show that

(i) $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

(ii) $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

(iii) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$

**Sol** (i) Let $z_1 = a+bi$ $\quad \bar{z}_1 = a-bi$

$z_2 = c+di$, $\quad \bar{z}_2 = c-di$

LHS
$$z_1+z_2 = a+bi+c+di$$
$$z_1+z_2 = (a+c)+(b+d)i$$
$$\overline{z_1+z_2} = (a+c)-(b+d)i \text{ —— (I)}$$
RHS
$$\bar{z}_1+\bar{z}_2 = a-bi+c-di$$
$$= (a+c)-(b+d)i \text{ —— (II)}$$
(I) = (II) $\Rightarrow$ $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$

(ii)

LHS
$$z_1 z_2 = (a+bi)(c+di)$$
$$= ac+adi+bci+bd(i^2)$$
$$= ac+adi+bci+bd(-1)$$
$$z_1 z_2 = (ac-bd)+(ad+bc)i$$
$$\overline{z_1 z_2} = (ac-bd)-i(ad+bc) \text{ —— (I)}$$
RHS
$$\bar{z}_1\bar{z}_2 = (a-bi)(c-di)$$
$$= ac-adi-bci+bd(i)^2$$
$$\bar{z}_1\bar{z}_2 = (ac-bd)-i(ad+bc) \text{ —— (II)}$$
(I) = (2) $\Rightarrow$ $\overline{z_1 z_2} = \bar{z}_1\bar{z}_2$

(iii)

LHS
$$\frac{z_1}{z_2} = \frac{a+bi}{c+di}$$
$$\frac{z_1}{z_2} = \frac{a+bi}{c+di}\times\frac{c-di}{c-di}$$
$$= \frac{ac-adi+bci-bd(i^2)}{c^2-(id)^2}$$
$$= \frac{ac-adi+bci-bd(-1)}{c^2+d^2}$$
$$\frac{z_1}{z_2} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$$
$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{(ac+bd)-i(bc-ad)i}{c^2+d^2} \text{ —— (I)}$$

Imp

**Th** Show that the modulus $|z|$ of a complex number $z = a+bi$ is the distance of a point from origin, or length of $\overline{OP}$.

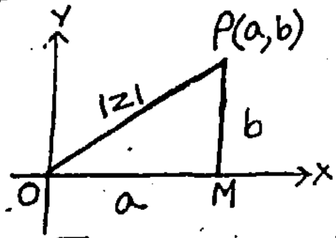
**Sol** We know to each complex number $z=a+bi$ there correspond a pt $P(a,b)$, in the cartesian plane and vice versa.

But the point $(a,b)$ in the plane is represented as

$|\overline{OM}| = a$

$|\overline{PM}| = b$

$|\overline{OP}| = |z|$




By pythagoras Th.
$$|\overline{OP}|^2 = |\overline{OM}|^2+|\overline{PM}|^2$$
$$|OP|^2 = a^2+b^2$$
$$|OP| = \sqrt{a^2+b^2}$$
$$|z| = \sqrt{a^2+b^2}$$
Distance of $(a,b)$ from $(0,0)$

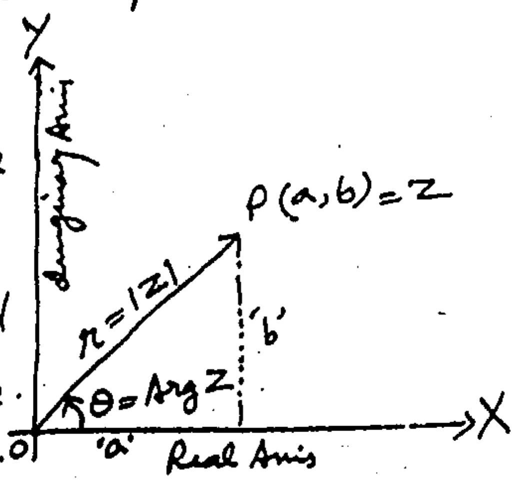
**Note 1)**
$$z = a+ib$$
$$\bar{z} = a-ib$$
$$\bar{\bar{z}} = a+ib \quad \therefore \boxed{\bar{\bar{z}} = z}$$

**Note 2)**
$$z = a+ib \Rightarrow |z| = \sqrt{a^2+b^2}$$
$$\bar{z} = a-ib \Rightarrow |\bar{z}| = \sqrt{a^2+b^2}$$
$$-z = -a-ib \Rightarrow |-z| = \sqrt{a^2+b^2}$$
$$\therefore |z| = |-z| = |\bar{z}|$$

RHS
$$\frac{\bar{z}_1}{\bar{z}_2} = \frac{a-bi}{c-di}$$
$$= \frac{a-bi}{c-di}\times\frac{c+di}{c+di}$$
$$= \frac{ac+adi-bci-bd(i^2)}{c^2-(id)^2}$$
$$= \frac{ac+adi-bci+bd}{c^2+d^2}$$
$$\frac{\bar{z}_1}{\bar{z}_2} = \frac{(ac+bd)-i(bc-ad)}{c^2+d^2} \text{ —— (II)}$$
(I) = (II) $\Rightarrow$ $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$

## Complex Plane

A complex number $z=a+bi$ corresponds to the pt $(a,b)$ in the XY-plane and vice versa.




The XY-plane in which a complex number $z$ is represented by a vector $\overline{OP}$ is called Complex Plane or Z-Plane, X-axis is called Real Axis and Y-axis is called Imaginary Axis & figure so obtained is called Argand Diagram.

The inclination '$\theta$' of a complex vector $\overline{OP}$ with positive direction of $x$-axis is called Argument or Amplitude of $z$, written as arg $z$.

$$\arg z=\theta=\tan^{-1}\frac{b}{a}$$

$$\sin\theta=\frac{b}{|z|}$$

$$\cos\theta=\frac{a}{|z|}$$

If value of $\theta$ is such that $-\pi<\theta\leq\pi$ then $\theta$ is called Principal argument of $z$ i.e Arg $z$

Arg of 0 is not defined.

### Imp Note

i) When $z=(a,b)$ is in Ist Quad then angle is '$\theta$'

ii) When $z=(a,b)$ is in 2nd Quad then angle is $(\pi-\theta)$

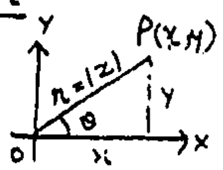
iii) When $z=(a,b)$ is in 3rd Quad then angle is $-(\pi-\theta)$

iv) When $z=(a,b)$ is in 4th Quad then angle is $-$'$\theta$'

It is because $-\pi<\theta\leq\pi$ i.e value of $\theta$ is not greater than $\pi$.

### Polar Form of Complex Number

From fig




$x=r\cos\theta,\ y=r\sin\theta$

$$z=x+iy \quad\text{——— (I)}$$

$$=r\cos\theta+i\,r\sin\theta$$

$$=r(\cos\theta+i\sin\theta)$$

$$z=r\,\text{Cis}\,\theta \quad\text{——— (II)}$$

(I) is Cartesian form of complex number $z$.

(II) is Polar form of complex number $z$.

### Properties

i) $|z|=|-z|=|\bar z|$

ii) $|z|^2=|z^2|=z\bar z$

iii) $\frac{1}{z}=\frac{\bar z}{|z|^2}$

iv) $|z_1z_2|=|z_1||z_2|$

v) $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}$

vi) $|\text{Re}\,z|\leq|z|$

vii) $|\text{Im}\,z|\leq|z|$

viii) $z\bar z=(\text{Re}\,z)^2+(\text{Im}\,z)^2$

ix) $|z_1-z_2|=|z_2-z_1|$

x) $|z_1-z_2|\geq|z_1|-|z_2|$

xi) $|z_1+z_2|\leq|z_1|+|z_2|$

**Th** For all $z_1, z_2 \in C$

$$\boxed{|z_1 z_2| = |z_1||z_2|}$$

Proof: Let $z_1 = a+ib$, $z_2 = c+id$, then

$$z_1z_2 = (a+ib)(c+id)$$
$$z_1z_2 = ac + iad + ibc + i^2 bd$$
$$z_1z_2 = (ac-bd) + i(ad+bc)$$
$$|z_1z_2| = \sqrt{(ac-bd)^2 + (ad+bc)^2}$$
$$= \sqrt{a^2c^2 + b^2d^2 - 2acbd + a^2d^2 + b^2c^2 + 2adbc}$$
$$= \sqrt{a^2(c^2+d^2) + b^2(d^2+c^2)}$$
$$= \sqrt{(a^2+b^2)(c^2+d^2)}$$
$$= \sqrt{a^2+b^2}\,\sqrt{c^2+d^2}$$
$$|z_1z_2| = |z_1||z_2| \quad \text{proved}$$

**Th** Prove that for $z_1, z_2 \in C$

$$\boxed{|z_1|-|z_2| \le |z_1+z_2| \le |z_1|+|z_2|}$$

Proof Let $|z_1+z_2|^2$

$= (z_1+z_2)\overline{(z_1+z_2)}$ ∵ $|z|^2 = z\bar{z}$

$= (z_1+z_2)(\bar{z}_1 + \bar{z}_2)$ ∵ $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$

$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$ ∵ $z_1\bar{z}_2 + z_2\bar{z}_1 = 2Re\, z_1\bar{z}_2$

$= |z_1|^2 + 2Re(z_1\bar{z}_2) + |z_2|^2$ ∵ $|Re\, z| \le |z|$

$\le |z_1|^2 + 2|z_1\bar{z}_2| + |z_2|^2$ ∵ $|z_1z_2| = |z_1||z_2|$, $|\bar{z}| = |z|$

$= |z_1|^2 + 2|z_1||\bar{z}_2| + |z_2|^2$

$= |z_1|^2 + 2|z_1||z_2| + |z_2|^2$

$|z_1+z_2|^2 \le (|z_1|+|z_2|)^2$

$\boxed{|z_1+z_2| \le |z_1| + |z_2|}$ —— (I)

Now $|z_1| = |z_1+z_2-z_2|$ $(+z_2-z_2)$

$\le |z_1+z_2| + |-z_2|$

$|z_1| = |z_1+z_2| + |z_2|$

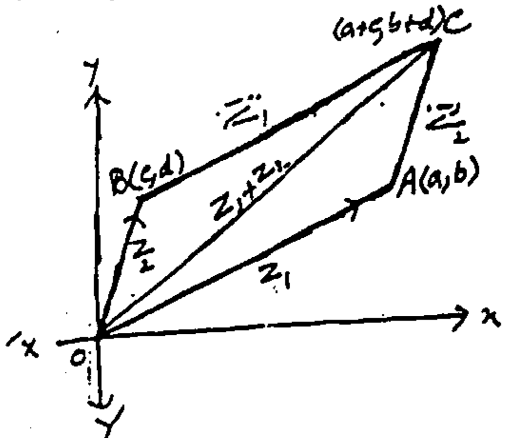
$\boxed{|z_1|-|z_2| \le |z_1+z_2|}$ —— (II)

Combining (I) & (II) $|z_1|-|z_2| \le |z_1+z_2| \le |z_1|+|z_2|$

**2nd Method** (Also see on Page 10)

For any two complex numbers

$$|z_1|-|z_2| \le |z_1+z_2| \le |z_1|+|z_2|$$




Let $|z_1| = |\overline{OA}|$ $\quad |z_1+z_2| = |\overline{OC}|$

$|z_2| = |\overline{OB}|$

In $\Delta OAC$ $\quad |\overline{OA}| + |\overline{AC}| > |\overline{OC}|$ —— (I)

$|z_1|+|z_2| > |z_1+z_2|$

For collinear pts

$|\overline{OA}| + |\overline{AC}| = |\overline{OC}|$

$|z_1|+|z_2| = |z_1+z_2|$ —— (II)

Combining (I) & (II)

$|z_1|+|z_2| \ge |z_1+z_2|$ proved

Now $|z_1| = |z_1+z_2-z_2|$

$|z_1| \le |z_1+z_2| + |-z_2|$

$|z_1| \le |z_1+z_2| + |z_2|$

$|z_1|-|z_2| \le |z_1+z_2|$ —— (III)

Combining (II) + (III) we get the result.

**To Prove** $|z_1|-|z_2| \le |z_1-z_2|$

Proof Since $|z_1+z_2| \le |z_1|+|z_2|$

Replace $z_1$ by $z_1-z_2$

∴ $|z_1-z_2+z_2| \le |z_1-z_2| + |z_2|$

⇒ $|z_1| \le |z_1-z_2|+|z_2|$

$|z_1|-|z_2| \le |z_1-z_2|$

**To Prove** $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$

Proof ∵ $z\bar{z} = |z|^2$

⇒ $\frac{1}{z\bar{z}} = \frac{1}{|z|^2}$

⇒ $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$ proved.

# EXERCISE 1.1

EXPRESS each of the following complex numbers in the polar form. (Problem 1—6):

**Q.1** Let $z = x + iy = -\sqrt{3} + i \Rightarrow x = -\sqrt{3}, \; y = 1$, $\quad r = |z| = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{3+1} = 2$

$\because \cos\theta = \frac{x}{r} = \frac{-\sqrt{3}}{2} \Rightarrow \theta = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$\& \sin\theta = \frac{y}{r} = \frac{1}{2} \Rightarrow \theta = \sin^{-1}\left(\frac{1}{2}\right)$ $\Big\} \Rightarrow \theta = \frac{5\pi}{6}$

($x$, is -ive & $y$, is +ive, So $\theta$ lies in 2nd Quad. Since 2nd Quadrant, $\therefore \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$)

Hence $z = r(\cos\theta + i\sin\theta)$
$= 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$
$= 2\,\text{cis}\,\frac{5\pi}{6}$ Ans

**2nd Method**

$z = x + iy = -\sqrt{3} + i \Rightarrow x = -\sqrt{3}, \; y = 1$

$\tan\theta = \frac{y}{x} = \frac{1}{-\sqrt{3}}$

$\theta = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) = \frac{5\pi}{6}$

($\because$ $x$ -ive, $y$ +ive, $\therefore$ $\theta$ lies in 2nd Quad. So Principal Arg $= \pi - \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$)

Hence $z = r(\cos\theta + i\sin\theta)$
$= 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

**Q.2** Let $z = x + iy = -i = 0 + (-i) \Rightarrow x = 0, \; y = -1$

$\Rightarrow r = |z| = \sqrt{0^2 + (-1)^2} = 1$

$\because \cos\theta = \frac{x}{r} = \frac{0}{1} = 0 \Rightarrow \theta = \cos^{-1}(0)$

$\& \sin\theta = \frac{y}{r} = \frac{-1}{1} = -1 \Rightarrow \theta = \sin^{-1}(-1)$ $\Big\} \theta = -\frac{\pi}{2}$

($\because$ $x$ +ive & $y$ -ive $\therefore$ 4th Quad. So Principal Arg $z = -\theta = -\frac{\pi}{2}$)

Hence $z = r(\cos\theta + i\sin\theta)$
$= 1\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$

or $z = \left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)$

**2nd Method**

$z = x + iy = -i$
$\Rightarrow x = 0, \; y = -1$

$\tan\theta = \frac{y}{x} = \frac{-1}{0} = \infty$

$\theta = \tan^{-1}(\infty) = -\frac{\pi}{2}$

($\because$ $x$ +ive, $y$ -ive $\therefore$ 4th Quad. So Principal Arg $= -\theta = -\frac{\pi}{2}$)

Hence $z = r(\cos\theta + i\sin\theta)$
$= 1\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$
$z = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$

**Q.3** Let $z = x + iy = -1 - \sqrt{3}i \Rightarrow x = -1, \; y = -\sqrt{3}$

$\Rightarrow r = |z| = \sqrt{(-1)^2 + (-\sqrt{3})^2}$

$\Rightarrow r = \sqrt{1+3} = \sqrt{4} = 2$

$\because \cos\theta = \frac{-1}{2} \Rightarrow \theta = \cos^{-1}\left(-\frac{1}{2}\right)$

$\& \sin\theta = \frac{-\sqrt{3}}{2} \Rightarrow \theta = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ $\Big\} \theta = -\frac{2\pi}{3}$

(3) $z = x + iy = -1 - \sqrt{3}i \Rightarrow x = -1, \; y = -\sqrt{3}$

$\tan\theta = \frac{y}{x} = \frac{-\sqrt{3}}{-1}$ $\quad r = \sqrt{1+3} = \sqrt{4}, \; r = 2$

$\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right)$

$\theta = -\frac{2\pi}{3}$

$\because$ $x$ -ive, $y$ -ive III Quad

$\therefore -\left(\pi - \frac{\pi}{3}\right) = -\frac{2\pi}{3}$

$\because$ $x$ is $-$ve $\therefore \theta$ is in 3rd Quad.
$y$ is $-$ve
So Principal $\text{Arg}\, z = -(\pi - \theta)$
Principal $\text{Arg}\, z = -(\pi - \frac{\pi}{3})$
$= -\frac{2\pi}{3}$

Hence $z = r(\cos\theta + i\sin\theta)$
$z = 2\left(\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right)$
$= 2\left(\cos\left(\frac{2\pi}{3}\right) - i\sin\left(\frac{2\pi}{3}\right)\right)$

Q.4 Let $z = x + iy = -1 + i \Rightarrow x = -1,\ y = 1$

$\Rightarrow r = |z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\cos\theta = \frac{x}{r} = \frac{-1}{\sqrt{2}} \Rightarrow \theta = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$
$\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$
$\theta = \frac{3\pi}{4} \therefore \theta = \frac{3\pi}{4}.$

$\because$ $x$ is $-$ve
$y$ is $+$ve $\therefore \theta$ is in 2nd Quad
Principal $\text{Arg}\, z = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Hence $z = r(\cos\theta + i\sin\theta)$
$= \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$
$= \sqrt{2}\ \text{Cis}\,\frac{3\pi}{4}$ Ans

Q4 2nd Method
$z = x + iy = -1 + i$ $\quad x = -1,\ y = 1$
$r = \sqrt{1+1} = \sqrt{2}$
$\tan\theta = \frac{y}{x} = \frac{1}{-1} = -1$
$\theta = \tan^{-1}(-1) = \frac{3\pi}{4}$
$x$ $-$ve, $y$ $+$ve II Quad
$\therefore \pi - \theta$
$\pi - \frac{\pi}{4} = \frac{3\pi}{4}$
$\therefore z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\,S\,\frac{3\pi}{4}\right)$

Q5 Let $z = (-2 + 2i)(1 - i)$
$= -2 + 2i + 2i - 2i^2$
$= \cancel{-2} + 4i + \cancel{2}$
$z = 4i$
$z = 0 + 4i \Rightarrow x = 0,\ y = 4$
$r = |z| = \sqrt{0^2 + 4^2} = 4$
$\cos\theta = \frac{x}{r} = \frac{0}{4} = 0 \Rightarrow \theta = \frac{\pi}{2}$
$\sin\theta = \frac{y}{r} = \frac{4}{4} = 1 \Rightarrow \theta = \frac{\pi}{2}$
($x$ is $+$ve, $y$ is $+$ve) $\theta$ is in Ist Quad So Principal $\text{Arg}\, z = \frac{\pi}{2}$)
Hence $z = r(\cos\theta + i\sin\theta) = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 4\,\text{Cis}\,\frac{\pi}{2}$ Ans.

Q5 $z = (-2 + 2i)(1 - i)$
$= 4i$
$z = x + iy = 4i$
$x = 0,\ y = 4$
$r = \sqrt{0^2 + 4^2}$, $r = 4$
$\tan\theta = \frac{y}{x} = \frac{4}{0} = \infty$
$\theta = \tan^{-1}(\infty)$
$\theta = \frac{\pi}{2}$
$\because x$ $+$ve, $y$ $+$ve I Quad
$\therefore z = 4\left(\cos\frac{\pi}{2} + i\,S\,\frac{\pi}{2}\right)$

**Q6** Let $z = -\dfrac{34i}{5-3i}$

$= \dfrac{-34i}{5-3i} \times \dfrac{5+3i}{5+3i}$

$= \dfrac{-34i(5+3i)}{25+9}$

$= -i(5+3i)$

$= -5i+3 \quad \Rightarrow x = 3, \; y = -5$

$r = \sqrt{3^2+(-5)^2} = \sqrt{34}$

$\theta = \tan^{-1}\dfrac{y}{x} = \tan^{-1}\left(\dfrac{-5}{3}\right)$

$\because x$ +ive, $y$ −ive $\therefore \theta$ is in IV$^{th}$ Q $\therefore$ Principal Arg $= -\theta$

So Principal Arg $z = -\tan^{-1}\left(\dfrac{5}{3}\right)$

$= \tan^{-1}\left(-\dfrac{5}{3}\right)$

$z = r(\cos\theta + i\sin\theta)$

$= \sqrt{34}\left(\cos\left(\tan^{-1}\left(-\tfrac{5}{3}\right)\right) + i\sin\left(\tan^{-1}\left(-\tfrac{5}{3}\right)\right)\right)$

$= \sqrt{34}\ \text{cis}\left(\tan^{-1}\left(-\tfrac{5}{3}\right)\right)$

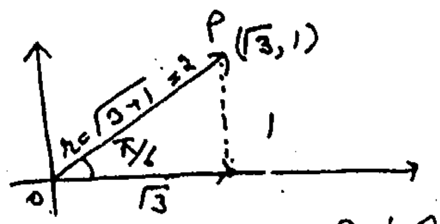
**Q7** Express the given Complex number in Cartesian form and in Argand Diagram.

$z = 2\,\text{cis}\left(\dfrac{\pi}{6}\right)$

$= 2\left[\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right]$

$= 2\left(\dfrac{\sqrt{3}}{2} + i\dfrac{1}{2}\right)$

$= \sqrt{3} + i \quad = (\sqrt{3}, 1)$




$\because x$ is +ive & $y$ is +ive so I st Quad.

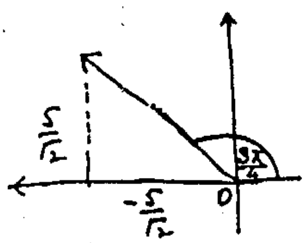
**Q8** $z = 5\,\text{cis}\left(\dfrac{3\pi}{4}\right)$

$= 5\left[\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right]$

$= 5\left[-\dfrac{1}{\sqrt{2}} + i\dfrac{1}{\sqrt{2}}\right]$

$= \dfrac{-5}{\sqrt{2}} + i\dfrac{5}{\sqrt{2}}$

$= \left(-\dfrac{5}{\sqrt{2}}, \dfrac{5}{\sqrt{2}}\right)$




$\frac{3\pi}{4} = 135°$

$\because x$ is −ive, $y$ is +ive so II nd Q

**Q9** $z = \sqrt{3}\,\text{Cis}\,\frac{7\pi}{6}$

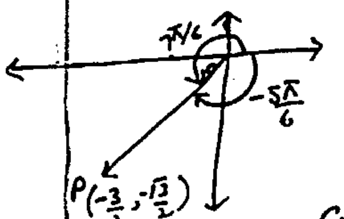
$= \sqrt{3}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$

$= \sqrt{3}\left(\cos\left(2\pi - \frac{5\pi}{6}\right) + i\sin\left(2\pi - \frac{5\pi}{6}\right)\right)$

$= \sqrt{3}\left(\cos\frac{5\pi}{6} - i\sin\frac{\pi}{6}\right)$

$= \sqrt{3}\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$

$= -\frac{3}{2} - i\frac{\sqrt{3}}{2} \qquad = \left(-\frac{3}{2}, -\frac{\sqrt{3}}{2}\right)$




Note: both are equal
$\boxed{\left(\pi + \frac{\pi}{6}\right) = \frac{7\pi}{6} = \left(2\pi - \frac{5\pi}{6}\right)}$

$\frac{7\pi}{6} - 2\pi = -\frac{5\pi}{6}$

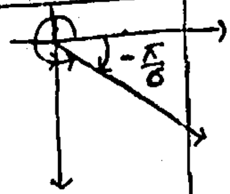
$\frac{5\pi}{6} = 150°$

$\cos\left(2\pi - \frac{5\pi}{6}\right) = \cos\frac{5\pi}{6}$

$\sin\left(2\pi - \frac{5\pi}{6}\right) = -\sin\frac{5\pi}{6}$

$x$ is −ve, $y$ is −ve so III rd Quad.
$\cos^{-1}\frac{\sqrt{3}}{2} = \boxed{\frac{\pi}{6}}$, $\sin^{-1}\frac{1}{2} = \boxed{\frac{\pi}{6}}$ So $-(\pi - \theta)$
$-\left(\pi - \frac{\pi}{6}\right) = -\frac{5\pi}{6}$

**Q10** $z = \dfrac{5\,\text{Cis}\left(\frac{\pi}{3}\right)}{2\,\text{Cis}\left(\frac{\pi}{2}\right)}$




$= \dfrac{5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}{2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)}$

$= \dfrac{5\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)}{2(0 + i\cdot 1)}$

$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$x \,\&\, \div$ by $i$ $\quad = \dfrac{5\cdot\left(\frac{i}{2} - \frac{\sqrt{3}}{2}\right)}{-2}$

$= \frac{5}{2}\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = \frac{5\sqrt{3}}{4} - \frac{5}{4}i = \left(\frac{5\sqrt{3}}{4}, -\frac{5}{4}\right)$

2nd Method

$z = \dfrac{5\,\text{Cis}\left(\frac{\pi}{3}\right)}{2\,\text{Cis}\left(\frac{\pi}{2}\right)}$

$= \frac{5}{2}\,\text{Cis}\left(\frac{\pi}{3} - \frac{\pi}{2}\right)$

$= \frac{5}{2}\,\text{Cis}\left(-\frac{\pi}{6}\right)$

$= \frac{5}{2}\left[\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right]$

$= \frac{5}{2}\left[\frac{\sqrt{3}}{2} - i\frac{1}{2}\right]$

$= \frac{5}{4}(\sqrt{3} - i)$

$x$ +ve, $y$ −ve So IV th Q.
So $-\theta$
So $-\frac{\pi}{6}$

**Q11 (i)**

Find $|z|$ where $z = -2i(1+i)(2+4i)(3+i)$

$|z| = |-2i(1+i)(2+4i)(3+i)|$

$= |-2i|\,|1+i|\cdot|2+4i|\cdot|3+i| \qquad \because |z_1 z_2| = |z_1|\cdot|z_2|$

$= \sqrt{(-2)^2}\sqrt{1^2+1^2}\sqrt{2^2+4^2}\sqrt{3^2+1^2}$

$= 2\sqrt{2}\sqrt{20}\sqrt{10}$

$= 2\sqrt{2}\,2\sqrt{5}\,\sqrt{5}\sqrt{2}$

$= 4(2)(5) \qquad = 40$

(ii) $z = \dfrac{(3+4i)(-1+2i)}{(-1-i)(3-i)}$

$|z| = \left|\dfrac{(3+4i)(-1+2i)}{(-1-i)(3-i)}\right|$

$= \dfrac{|(3+4i)(-1+2i)|}{|(-1-i)(3-i)|} \qquad \because \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

$|z| = \dfrac{|3+4i|\cdot|-1+2i|}{|-1-i|\,|3-i|}$

$= \dfrac{\sqrt{9+16}\;\sqrt{1+4}}{\sqrt{1+1}\;\sqrt{9+1}}$

$= \dfrac{5\;\sqrt{5}}{\sqrt{2}\;\sqrt{10}}$

$= \dfrac{5\sqrt{5}}{2\sqrt{5}}$

$= \frac{5}{2}$ Ans.

Imp **Q12(i)** Show that $z = a + ib$ is real iff $\boxed{z = \bar{z}}$

Let $z = a + ib$ is real $\Rightarrow b = 0$ (Imaginary part $b=0$)

$\therefore z = a$ ——(i)

$\bar{z} = a$ ——(ii)

from (i) & (ii) $\therefore \boxed{z = \bar{z}}$

**Conversely** Suppose $z = \bar{z}$

$a + ib = \overline{a + ib}$

$a + ib = a - ib$

$\cancel{a} - \cancel{a} + ib + ib = 0$

$2ib = 0$

$b = 0$ $\because \left( i = \sqrt{-1} \neq 0, \ 2 \neq 0 \right.$

Hence $z = a + 0i$

$z = a$ which is real

Imp **Q12 (ii)** Show that $z = a + ib$ is pure imaginary iff $\boxed{z = -\bar{z}}$

Let $z = a + ib$ is pure imaginary $\Rightarrow a = 0$ (Real part $a=0$)

So $z = ib$ ——(1)

$\bar{z} = -ib$ ——(2)

$\bar{z} = -(z)$ using (1) in (2)

$\boxed{z = -\bar{z}}$

Conversely Let $z = -\bar{z}$

$a + ib = -(a - ib)$

$a + \cancel{ib} + a - \cancel{ib} = 0$

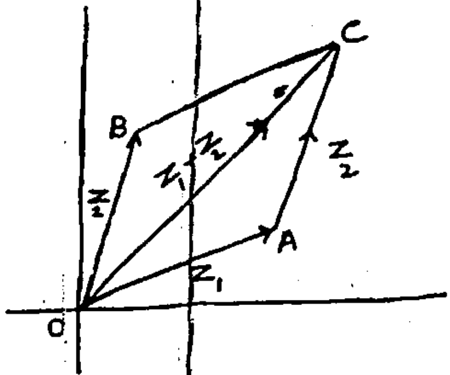
$2a = 0$

$a = 0$ (Real part of z is zero)

So $z = 0 + ib$

$z = ib$ which is pure imaginary.

**Example 3** Let $z_1, z_2$ be two complex numbers. Determine the greatest and least values of $|z_1+z_2|$



Sol Let $z_1 = \overline{OA}$

& $z_2 = \overline{OB}$

then $z_1+z_2 = \overline{OC}$

Now $|z_1| = |\overline{OA}|$

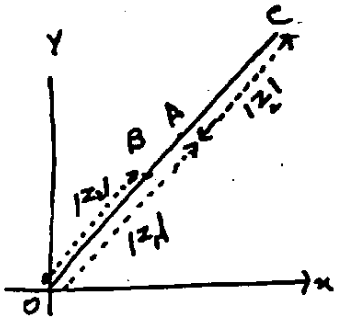
$|z_2| = |\overline{OB}| = |\overline{AC}|$

& $|z_1+z_2| = |\overline{OC}|$

In $\triangle OAC$

$|\overline{OA}| + |\overline{AC}| > |\overline{OC}|$ ($\because$ Sum of two sides lengths is greater than the length of third side in any $\triangle$)

$|z_1|+|z_2| > |z_1+z_2|$ ——— (I)



when $\operatorname{Arg} z_1 = \operatorname{Arg} z_2$

then OA is ∥ to OB

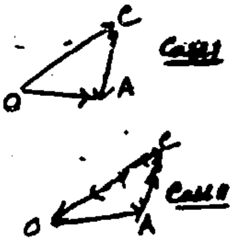
hence ∥gm OABC becomes straight line

$\therefore \quad OA + AC = OC$

$|z_1| + |z_2| = |z_1+z_2|$ ——— (II)

from (I) & (II) $|z_1|+|z_2| \geqslant |z_1+z_2|$ ——— (III)

Thus greatest possible value of $|z_1+z_2|$ is $|z_1|+|z_2|$



In $\triangle OAC$ $OC+CA > OA$ and $CO + OA > CA$

$|z_1+z_2|+|z_2| > |z_1|$ $\qquad$ $|z_1+z_2| + |z_1| > |z_2|$

$|z_1+z_2| > |z_1|-|z_2|$ ——— (IV) $\qquad$ $|z_1+z_2| > |z_2| - |z_1|$

$|z_1+z_2| > -(|z_1|-|z_2|)$

$-|z_1+z_2| < |z_1|-|z_2|$ ——— (V)

from (IV) & (V) $-|z_1+z_2| < |z_1|-|z_2| < |z_1+z_2|$ ——— (VI)

& (VI) together with the extreme case when O, A, B, C are collinear gives

$$\big||z_1|-|z_2|\big| \leqslant |z_1+z_2| \text{ ——— (VII)} \qquad \because \begin{pmatrix} -a \le x \le a \\ \Rightarrow |x| \le a \end{pmatrix}$$

Thus least possible value of $|z_1+z_2|$ is $\big||z_1|-|z_2|\big|$

Combining (III) & (VII)

$$\big||z_1|-|z_2|\big| \leqslant |z_1+z_2| \leqslant |z_1|+|z_2|$$

x ——— x

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Q13 Prove analytically for complex No $z_1, z_2$

$$\big||z_1|-|z_2|\big| \leq |z_1-z_2| \leq |z_1+z_2| \leq |z_1|+|z_2|$$

Sol Let

$$|z_1+z_2|^2 = (z_1+z_2)(\overline{z_1+z_2}) \qquad \because |z|^2 = z\bar{z}$$

$$= (z_1+z_2)(\bar{z}_1+\bar{z}_2) \qquad \because \overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$$

$$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$$

$$= |z_1|^2 + 2Re(z_1\bar{z}_2) + |z_2|^2 \qquad \because z_1\bar{z}_2 + z_2\bar{z}_1 = 2Re\, z_1\bar{z}_2 \text{ proved earlier}$$

$$\leq |z_1|^2 + 2|z_1\bar{z}_2| + |z_2|^2 \qquad \because |Re\, z| \leq |z|$$

$$= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \qquad \because |z_1 z_2| = |z_1||z_2| \ \& \ |\bar{z}_1| = |z_1|$$

$$|z_1+z_2|^2 \leq (|z_1|+|z_2|)^2$$

Taking Square Root

$$\boxed{|z_1+z_2| \leq |z_1|+|z_2|} \quad \text{——— (I)}$$

Now

$$|z_1| = |z_1+z_2-z_2| \qquad (+\ \&\ -z_2)$$

$$\leq |z_1+z_2| + |-z_2| \qquad \because |z_1+z_2| \leq |z_1|+|z_2|$$

$$= |z_1+z_2| + |z_2| \qquad \because |z_2| = |-z_2|$$

$$\boxed{|z_1|-|z_2| \leq |z_1+z_2|} \quad \text{——— (II)}$$

Put $z_2 = -z_2$ in (II)

$$|z_1|-|-z_2| \leq |z_1-z_2|$$

$$\boxed{|z_1|-|z_2| \leq |z_1-z_2|} \quad \text{——— (III)}$$

Also

$$|z_2| = |z_2-z_1+z_1| \qquad (+\ \&\ -z_1)$$

$$\leq |z_2-z_1| + |z_1|$$

$$|z_2|-|z_1| \leq |z_2-z_1|$$

$$|z_2|-|z_1| \leq |z_1-z_2| \qquad \because |z_1-z_2| = |z_2-z_1|$$

$$-|z_1-z_2| \leq |z_1|-|z_2| \quad \text{——— (IV)}$$

from (III) & (IV) $\therefore$

$$-|z_1-z_2| \leq |z_1|-|z_2| \leq |z_1-z_2|$$

$$\therefore \big||z_1|-|z_2|\big| \leq |z_1-z_2| \quad \text{——— (V)}$$

as if $-a \leq x \leq a$ then $|x| \leq a$

Now Obviously $|z_1-z_2| \leq |z_1+z_2|$ ——— (VI)

from (I) & (VI) $|z_1-z_2| \leq |z_1+z_2| \leq |z_1|+|z_2|$ ——— (VII)

from (V) & (VII) $\big||z_1|-|z_2|\big| \leq |z_1-z_2| \leq |z_1+z_2| \leq |z_1|+|z_2|$ Proved.

$z_1 = 24+7i$, $|z_2| = 6$

$|z_1| = \sqrt{(24)^2+7^2}$
$= \sqrt{576+49}$
$= \sqrt{625} = 25$

We know
$$\big||z_1|-|z_2|\big| \le |z_1+z_2| \le |z_1|+|z_2|$$
So greatest value of $|z_1+z_2|$ is
$$= |z_1|+|z_2| = 25+6 = 31$$
also since $\big||z_1|-|z_2|\big| \le |z_1+z_2|$

so least value of $|z_1+z_2|$ is
$$= \big||z_1|-|z_2|\big|$$
$$= |25-6| = |19| = 19$$

**Q.15** If $z_1, z_2$ are complex numbers, show that
$$|z_1+z_2|^2+|z_1-z_2|^2 = 2\left(|z_1|^2+|z_2|^2\right)$$

**PROOF** L.H.S. $|z_1+z_2|^2+|z_1-z_2|^2$

$= (z_1+z_2)(\overline{z_1+z_2})+(z_1-z_2)(\overline{z_1-z_2})$ $\quad (\because z\bar{z} = |z|^2)$

$= (z_1+z_2)(\bar{z}_1+\bar{z}_2)+(z_1-z_2)(\bar{z}_1-\bar{z}_2)$

$= z_1\bar{z}_1+z_2\bar{z}_2+z_1\bar{z}_2+z_2\bar{z}_1+z_1\bar{z}_1+z_2\bar{z}_2-z_1\bar{z}_2-z_2\bar{z}_1$

$= 2z_1\bar{z}_1+2z_2\bar{z}_2 - 2(z_1\bar{z}_1+z_2\bar{z}_2)$

$= 2\left(|z_1|^2+|z_2|^2\right) = $ R.H.S.

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**Q.16** Prove that $\left|\dfrac{az+b}{\bar{b}z+\bar{a}}\right| = 1$ for $|z|=1$

**Sol** we have L.H.S. $= \left|\dfrac{az+b}{\bar{b}z+\bar{a}}\right| = \dfrac{|az+b|}{|\bar{b}z+\bar{a}|}$ $\quad \left(\because \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}\right)$

$(\because |z| = |\bar{z}|)$

$= \dfrac{|az+b|}{|\overline{\bar{b}z+\bar{a}}|} = \dfrac{|az+b|}{|\bar{\bar{b}}\bar{z}+\bar{\bar{a}}|}$

$= \dfrac{|az+b|}{|\bar{\bar{b}}\bar{z}+\bar{\bar{a}}|} = \dfrac{|az+b|}{|b\bar{z}+a|}$ $\quad (\because \bar{\bar{z}} = z)$

$= \dfrac{|z|\,|az+b|}{|z|\,|b\bar{z}+a|} = \dfrac{|z|\,|az+b|}{|bz\bar{z}+az|}$ $\quad (\because |z_1z_2| = |z_1||z_2|)$

$$= \frac{|z||az+b|}{|az+b|} \quad \left(\because \; z\bar{z} = |z|^2 = 1 \text{ as } |z| = 1 \text{ given}\right)$$

$$= |z| = 1 = \text{R.H.S}$$

**Q.17** Find locus of the points in the plane which satisfying each of the following conditions:

**Part-(i)** $|z-5| = 6$

**Sol** $|z-5| = 6 \rightarrow (i)$ let $z = a+ib$

Then (i) will become

$|a+ib-5| = 6$ or $|(a-5)+ib| = 6$

or $\sqrt{(a-5)^2 + b^2} = 6$ or $(a-5)^2+(b-0)^2 = 36$

which shows that the locus is a circle having centre at point $(5,0)$ and radius $= 6$ unit

**Part-(ii)** $|z-2i| \geq 1$

**Sol** we have $|z-2i| \geq 1 \longrightarrow (i)$

let $z = x+iy$. So (i) will be

$|x+iy-2i| \geq 1$ or $|x+i(y-2)| \geq 1$

or $\sqrt{x^2+(y-2)^2} \geq 1$ or $x^2+(y-2)^2 \geq 1$

or $(x-0)^2+(y-2)^2 \geq 1 \longrightarrow (ii)$

inequality give that required locus is a set of those points that lies on the circle or outside the circle having centre at $(0,2)$ and radius $=1$

**Part-(iii)** $Re(z+2) = -1$

**Sol** We are given that

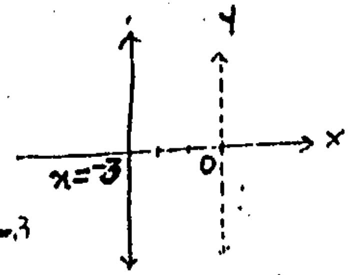
$Re(z+2) = -1 \longrightarrow (i)$

let $z = x+iy$, put in (i)

$\Rightarrow \quad Re\,(x+iy+2) = -1$

or $\quad Re\,((x+2)+iy) = -1$

$\Rightarrow \quad x+2 = -1 \quad$ or $\quad x = -3$




The locus is the line $\parallel$ to Y-axis on left side of y-axis.

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**Part - (iv)** $\quad Re(i\bar{z}) = 3$

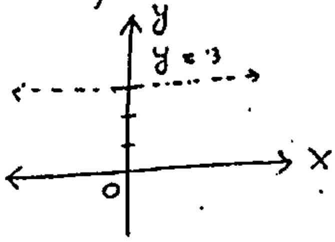
Sol. We have $\quad Re\,(i\,\bar{z}) = 3 \longrightarrow$ (i)

let $\quad z = x+iy \quad$ then (i) will be

$Re\,(i\,\overline{x+iy}) = 3 \quad$ or $\quad Re\,(i\,(x-iy)) = 3$

or $\quad Re\,(y+ix) = 3 \quad \Rightarrow \quad y = 3$

$\Rightarrow$ loucus is the horizontal line $y=3$.

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**Part - (v)** $\quad |z+i| = |z-i| \longrightarrow$ (i)

Sol. Put $z = x+iy$ in (i) we get

$|x+iy+i| = |x+iy-i|$

or $\quad |x+i(y+1)| = |x+i(y-1)|$

$\Rightarrow \quad \sqrt{x^2+(y+1)^2} = \sqrt{x^2+(y-1)^2}$

Sq. we get

$x^2+(y+1)^2 = x^2+(y-1)^2$

or $\quad (y+1)^2 = (y-1)^2$

$\Rightarrow \quad y^2+2y+1 = y^2-2y+1 \quad \Rightarrow \quad 2y = -2y$

or $\quad 4y = 0 \quad \Rightarrow \quad y = 0 \longrightarrow$ (ii)

Eq. (ii) gives the required locus is set of all those points that lies on x-axis.

## Part-(vi) $|z+3| + |z+1| = 4 \rightarrow (i)$

**Sol** Put $z = x+iy$ in (i) we get

$$|x+yi+3| + |x+iy+1| = 4$$

or $|(x+3)+iy| + |(x+1)+iy| = 4$

or $\sqrt{(x+3)^2+y^2} + \sqrt{(x+1)^2+y^2} = 4$

$\Rightarrow \sqrt{(x+3)^2+y^2} = 4 - \sqrt{(x+1)^2+y^2}$

Sq. we get

$$(x+3)^2+y^2 = 16 + (x+1)^2 + y^2 - 8\sqrt{(x+1)^2+y^2}$$

$$\cancel{x^2}+\cancel{y^2}+6x+9-\cancel{x^2}-2x-17-\cancel{y^2} = -8\sqrt{(x+1)^2+y^2}$$

$$4x-8 = -8\sqrt{(x+1)^2+y^2}$$

or $x-2 = -2\sqrt{(x+1)^2+y^2}$

Sq. we get

$$x^2-4x+4 = 4(x^2+2x+1+y^2)$$

or $3x^2+12x+4y^2=0$ which is required locus

x———x

## Part-(vii) $-1 \leq Re\,z < 1$ → (i)

**Sol** Put $x+iy = z$ in (i)

$\Rightarrow -1 \leq Re(x+iy) < 1$

or $-1 \leq x < 1$

$\Rightarrow$ The value of $x$ lies in the interval $[-1, 1[$.

x———x

## Part-(viii) $Im\,z < 0 \rightarrow (i)$

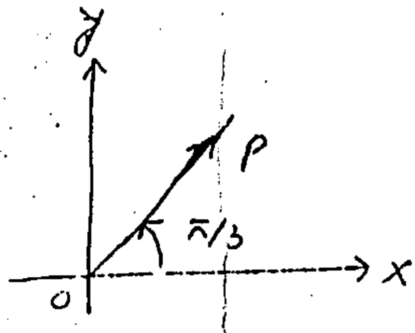
**Sol** Put $z = x+iy$ in (i)

$\Rightarrow Im(x+iy)<0$ or $y<0$ which is required locus i.e. value of $y$ is $-$ive.

**Part - (ix)** $\text{Arg } z = \frac{\pi}{3}$

<u>Sol</u> Let $z = \overrightarrow{OP}$. then $\text{Arg } z = \text{Arg } \overrightarrow{OP} = \frac{\pi}{3}$

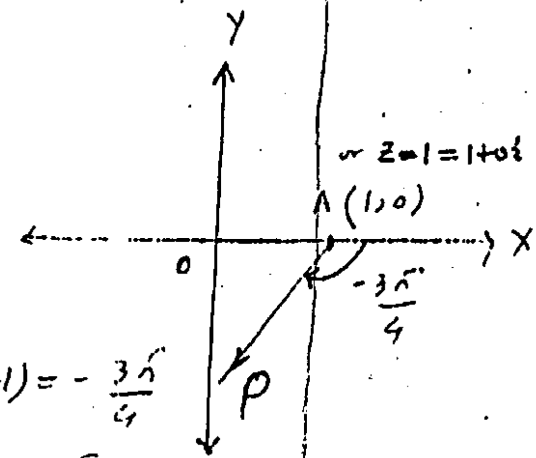
So, the required locus is a line $\overrightarrow{OP}$ that makes $\theta = \frac{\pi}{3}$ with +ive x-axis, as shown in fig




× ———— ×

## Part -(x)

$\text{Arg}(z-1) = -\frac{3\pi}{4}$




<u>Sol</u> put $z = x+iy$

$\Rightarrow \text{Arg}(z-1) = \text{Arg}(x+iy-1) = -\frac{3\pi}{4}$

or $\text{Arg}\left((x-1)+iy\right) = -\frac{3\pi}{4}$

or $\text{Arg}\left((x-1)+i(y-0)\right) = -\frac{3\pi}{4}$

The required locus is represented by the line $\overrightarrow{AP}$ which makes an angle of measure $= -\frac{3\pi}{4}$ with +ive x-axis at pt $A(1,0)$ on x-axis

× ———— ×

# DE MOIVRE'S THEOREM

**STATEMENT:-** If $n$ is any integer, then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \rightarrow (1)$$

**PROOF:-** CASE-1 Put $n=0$ in (1), then

$L.H.S = (\cos\theta + i\sin\theta)^0 = 1$

$R.H.S = \cos(0\cdot\theta) + i\sin(0\cdot\theta) = \cos 0 + i\sin 0 = 1+0i = 1$

$\Rightarrow L.H.S. = R.H.S.$

Chapter 9

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**Differential Equation:-** An eq involving independent and dependent variables and the derivatives of the dependent variable with respect to one or more independent variables is called a Diff Eq. In $\frac{dy}{dx}, \frac{d^2z}{du^2}$ $y, z$ are dependent variables & $x, u$ are independent variables.

**Ordinary Diff Eq:-** An eq involving only derivatives of one or more dependent variables, with respect to a single independent variable is called ordinary diff eq. e.g $\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0$ is O.D.Eq, but $\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dt}\right)^2 = 0$ is not O.D.Eq

**Partial Diff Eq:-** An eq involving partial derivatives of one or more dependent variables with respect to two or more independent variables is called partial diff eq.

**Order of Diff Eq:-** The order of a diff eq is the order of the highest derivative that occurs in the eq.

**Degree of Diff Eq:-** The degree of a diff eq is the power of highest order derivative (free of fractions) involved in a diff eq.

| | | | |
|---|---|---|---|
| (i) $\frac{dy}{dx} + y\cos x = \sin x$ | Ordinary Diff Eq | order 1 | Degree 1 |
| (ii) $\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0$ | " | order 2 | Degree 1 |
| (iii) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = \frac{d^2y}{dx^2}$ | " | order 2 | Degree 2 ☆ |
| (iv) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$ | Partial Diff Eq | order 1 | Degree 1 |
| (v) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ | " | order 2 | Degree 1 |
| (vi) $\frac{d^2v}{dx^2} + \frac{dv}{dx} = e^x$ | Ordinary Diff Eq | order 2 | Degree 1 |

☆ $\left(\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}\right)^2 = \left(\frac{d^2y}{dx^2}\right)^2 \Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = \left(\frac{d^2y}{dx^2}\right)^2$

Let $(y''')^{2/3} = 4 + y'$

cubing both sides $(y''')^2 = (4+y')^3$ — Order 3 degree 2.

$y''' = \sqrt{2x+3y}$ — Order 3 degree 1

$6x^2 \frac{d^3y}{dx^3} + \sin x \frac{d^2y}{dx^2} - \cos xy$

Degree is undefined ∵ the unknown fn 'y' is argument of transcendental cosine fn and therefore can not be written as a polynomial in y and its derivatives.

Similarly $y'' + (y')^2 = \log y''$

& $\sin\left(\frac{dy}{dx}\right) = \frac{dy}{dx} + 3x + 2$

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**Linear Diff Eq.**

A diff eq is said to be linear if

i) the dependent variable y and its derivatives are all of degree "one" only.

ii) No products of 'y' and its derivatives are present.

iii) No transcendental fn of y or its derivatives are present.

e.g. $2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 3y = 0$

$\frac{dy}{dx} - x^2 y = \cos x.$

A diff eq which is not linear is called Non-Linear Diff Eq:

eg

i) $\frac{d^3y}{dx^3} + 4y^2 = 0$ (power of $y \neq 1$)

ii) $\frac{d^2y}{dx^2} + 7y\frac{dy}{dx} + 12y = 0$ (∵ $7y\frac{dy}{dx}$ involves product of 'y' dependent variable & derivati

iii) $\frac{d^2y}{dx^2} + \sin xy = 0$ (involves transcendental fns of dependent variable.

iv) $5\left(\frac{dy}{dx}\right)^3 + 2\frac{d^2y}{dx^2} + 3y = 0$ (∵ degree of $\frac{dy}{dx}$ is not 1)

# Exercise 9.1

① Classify each of the following eqs as ordinary or partial diff eq state the order and degree of each eq and determine whether the eq is linear or non-linear.

(i) $\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 3y = \cos x$

Ordinary Diff Eq, order 3, degree 1,

It is Linear Diff Eq.

(ii) $x^2 dy + y^2 dx = 0 \Rightarrow \frac{dy}{dx} + \frac{y^2}{x^2} = 0$

Ordinary Diff Eq, order 1, degree 1

It is non linear eq $\because$ power of $y \neq 1$

(iii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

Partial Diff Eq, order 2, degree 1

It is Linear Diff Eq.

iv) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} + u = 0$

It is Partial Diff Eq. order 2, degree 1

Non-linear Diff Eq $\because u\frac{\partial u}{\partial x}, u\frac{\partial u}{\partial y}$ i.e. Product.

v) $\left(\frac{dy}{dx}\right)^2 = \left(\frac{d^2y}{dx^2} + y\right)^{3/2}$

Ordinary Diff Eq, order 2, Degree 3

$\because \left[\left(\frac{dy}{dx}\right)^2\right]^2 = \left[\left(\frac{d^2y}{dx^2} + y\right)^{3/2}\right]^2$

$= \left(\frac{d^2y}{dx^2} + y\right)^3$

Non linear Diff Eq $\because$ Degree $\neq 1$

General Solution or (Integral) or (Complete Primitive) :-

A sol of a diff eq which contains the number of arbitrary constants equal to the order of the diff eq is called General Sol.

Particular Solution :-

A sol obtained from the general sol by giving particular values to the constants is called a particular sol or integral.

Example: The general Sol of diff eq $\frac{d^2y}{dx^2}=0$ is $y=mx+c$. order 2 const 2, i.e $m, c$

whereas $y=3x+5$ is obtained by taking particular values $m=3$ & $c=5$.

Singular Sol :- (S.S)

A sol of a diff eq which cannot be obtained from the general sol by any choice of independent arbitrary const is called singular sol.

e.g the general sol of $y'=\sqrt{y}$ is $2\sqrt{y}=x+c$ and S.S is $y=0$

Note The arbitrary constants appearing in the general sol of a diff eq must be independent and to check this we show that they cannot be replaced by or reduced to a smaller number of const.

e.g $y = l\sin(x+\alpha)+m\cos x$ is the sol of $\frac{d^2y}{dx^2}+y=0$

it seems to contain three const $l, m, \alpha$. But they are not independent as they can be reduced to 'two' only.

$$y = l\sin(x+\alpha)+m\cos x$$
$$= l\sin x\cos\alpha + l\cos x\sin\alpha + m\cos x$$
$$=(l\cos\alpha)\sin x + (m+l\sin\alpha)\cos x$$

or $y = A\sin x + B\cos x$ so two Arbitrary independent Consts $A+B$

Initial Value Condition is a condition on the sol of a diff eq at one pt. i.e '$x_0$' e.g $y(x_0)=a$, $y'(x_0)=b$ i.e at $x=\boxed{x_0}$ $y=a$ & $y'=b$

Boundary Value Condition is a cond on the sol of a diff eq at more than one pt. i.e $x_0, x_1$ $y(x_0)=a$, $y(x_1)=b$

Formation of a differential Eq. A diff eq is formed by the elimination of arbitrary constants from a relation of the form $f(x,y)=0$.

Since to eliminate one const we need two eqs, and to eliminate two constants we need three eqs and so on. N

Now we shall be given one eq of the form $f(x,y)=0$ and the remaining required number of eqs will be formed by differentiating given eq the required number of times.

This also shows that the order of the required diff eq can not exceed the number of constants to be eliminated

Thus we shall not diff the given eq more than the number of consts. eg to form diff eq from

$$y^2 = cx \quad \text{——— (1)}$$

diff $2y\dfrac{dy}{dx} = c$ ——— (2)

Required diff eq will be obtained by eliminating 'c'. between (1) & (2)

So Put (2) in (1) $y^2 = x\left(2y\dfrac{dy}{dx}\right)$

Note As there is just one const, so the required diff eq is to be of order one'. i,e we should not diff (2) again to eliminate c as

Diff (1) $2y\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx}\cdot\dfrac{dy}{dx} = 0$ c eliminated

(8)

## Ex 9.1

**Q2** (i)

Form the diff eq of which the given fn is a sol.

(i) $y = x + 3e^{-x}$

Diff $y' = 1 - 3e^{-x}$

$= 1 - (y - x)$ $\quad \because y = x + 3e^{-x}$ eliminating $e$

$y' + y = x + 1$

×————×

(ii) $y = (x^3 + c)e^{-3x}$, c being arbitrary const.

diff $y' = 3x^2 e^{-3x} + (x^3+c)(-3e^{-3x})$

$= 3x^2 e^{-3x} + (-3)y$ $\quad \because y = (x^3+c)e^{-3x}$ C eliminated

$y' + 3y = 3x^2 e^{-3x}$

×————×

(iii) $ax + \ln|y| = y + b$

Diff $a + \frac{1}{y}y' = y'$

Diff $-y^{-2}y'y' + \frac{1}{y}y'' = y''$

$\frac{-(y')^2}{y^2} + \frac{y''}{y} = y''$

$\frac{-(y')^2 + yy''}{y^2} = y''$

$-(y')^2 + y\,y'' = y^2 y''$

$-(y')^2 + y''(y - y^2) = 0$

×————×

(iv) $y = ae^{x} + b\ln x + cx + d$ $\quad$ four const so diff four times

Diff $y' = ae^{x} + \frac{b}{x} + c$ —— (i)

Diff $y'' = ae^{x} - \frac{b}{x^2}$ —— (ii)

Diff $y''' = ae^{x} + \frac{2b}{x^3}$ —— (iii)

Diff $y^{iv} = ae^{x} - \frac{6b}{x^4}$ —— (iv)

Eliminating a & b from (ii) (iii) (iv)

$$e^{x}\, x^2 \begin{vmatrix} y'' & 1 & -1 \\ y''' & 1 & \frac{2}{x} \\ y^{iv} & 1 & \frac{-6}{x^2} \end{vmatrix} = 0$$

(v) $x^2+y^2+2gx+2fy+c=0$ three const g, f, c. So diff thrice

Diff $2x+2yy'+2g+2fy'=0$

$x+yy'+g+fy'=0$

$(x+g)+(y+f)y'=0$ — (1)

Diff $1+(y+f)y''+y'y'=0$ — (2)

Diff $(y+f)y'''+y'y''+[y'y''+y''y']=0$

$(y+f)y'''+3y'y''=0$ — (3)

$(y+f)=\frac{-3y'y''}{y'''}$ Put in (2)

from (2) $1+\left(\frac{-3y'y''}{y'''}\right)y''+y'^2=0$

$(1+y'^2)=\frac{3y'(y'')^2}{y'''}$

$(1+y'^2)y'''=3y'(y'')^2$

$3y'(y'')^2-(1+y'^2)y'''=0$

x———x

(vi) $u=f(x,y,z)=(x^2+y^2+z^2)^{-\frac{1}{2}}$ Diff thrice partially w.r.t x, y, z.

$\frac{\partial u}{\partial x}=-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x)$

$\frac{\partial u}{\partial x}=-\frac{x}{(x^2+y^2+z^2)^{3/2}}$

$\frac{\partial^2 u}{\partial x^2}=-\left(\frac{(x^2+y^2+z^2)^{3/2}-x\cdot\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}2x}{(x^2+y^2+z^2)^3}\right)$

$=-\left(\frac{(x^2+y^2+z^2)^{\frac{3}{2}}-3x^2(x^2+y^2+z^2)^{\frac{1}{2}}}{(x^2+y^2+z^2)^3}\right)$

$=-\frac{(x^2+y^2+z^2)^{\frac{1}{2}}\{x^2+y^2+z^2-3x^2\}}{(x^2+y^2+z^2)^3}$

$\frac{\partial^2 u}{\partial x^2}=-\frac{(-2x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}}=\frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$ — (i)

Similarly

$\frac{\partial^2 u}{\partial y^2}=\frac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}$ — (ii)

& $\frac{\partial^2 u}{\partial z^2}=\frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$ — (iii)

Adding $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$

x———x

(8)

vii) $u = f(x-ay) + g(x+ay)$ f & g are twice diff$^l$ fns.

$$\frac{\partial u}{\partial x} = f'(x-ay) + g'(x+ay)$$

$$\frac{\partial^2 u}{\partial x^2} = f''(x-ay) + g''(x+ay) \quad \text{——} (1)$$

$$\frac{\partial u}{\partial y} = f'(x-ay)(-a) + g'(x+ay)(a)$$

$$\frac{\partial^2 u}{\partial y^2} = f''(x-ay)(-a)(-a) + g''(x+ay)(a)(a)$$

$$= a^2\left[f''(x-ay) + g''(x+ay)\right]$$

$$\frac{\partial^2 u}{\partial y^2} = a^2 \frac{\partial^2 u}{\partial x^2} \quad \text{using (1)}$$

(3)$^{MT}$ (1) Find the differential eq of all circles of radius 'a'. (a is fixed)

Eq of circles of radius 'a': $(x-h)^2 + (y-k)^2 = a^2$

a is fixed given (h, k two arbitrary const so differentiate twice)

Diff $2(x-h) + 2(y-k)y' = 0$

$$(x-h) + (y-k)y' = 0 \quad \text{——} (1)$$

Diff $1 + (y-k)y'' + y'y' = 0$

$$(y-k)y'' = -1 - y'^2$$

$$(y-k) = \frac{-(1+y'^2)}{y''} \quad \text{——} (2)$$

Put (2) in (1) $(x-h) - \frac{(1+y'^2)}{y''}y' = 0$

$$(x-h) = \frac{(1+y'^2)}{y''}y' \quad \text{——} (3)$$

Squaring & Adding (2) & (3) to eliminate consts.

$$(x-h)^2 + (y-k)^2 = \left(\frac{1+y'^2}{y''}\right)^2 y'^2 + \left(\frac{1+y'^2}{y''}\right)^2$$

$$a^2 = \left(\frac{1+y'^2}{y''}\right)^2\left[y'^2 + 1\right]$$

$$a^2(y'')^2 = (1+y'^2)^2(y'^2+1)$$

$$a^2(y'')^2 = (1+y'^2)^3$$

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(ii) Find the diff eq of circles that pass through origin.

Eq of all circles passing through origin is

$x^2+y^2+2gx+2fy = 0$ — ① Two Const f, g So diff twice

Diff $2x+2yy'+2g+2fy' = 0$

$x+yy'+g+fy' = 0$

$(x+g)+y'(y+f) = 0$ — ②

Diff $1+(y+f)y''+y'y' = 0$

$(y+f) = -\frac{(1+y'^2)}{y''}$ — ③

Put ③ in ② $(x+g)+y'\left(-\frac{(1+y'^2)}{y''}\right) = 0$

$(x+g) = y'\left(\frac{1+y'^2}{y''}\right)$ — ④

Multiply ④ by $x$ & ③ by $y$ and adding

$x(x+g)+y(y+f) = xy'\left(\frac{1+y'^2}{y''}\right)-y\frac{(1+y'^2)}{y''}$

$x^2+gx+y^2+fy = (xy'-y)\left(\frac{1+y'^2}{y''}\right)$

$x^2+y^2+(gx+fy) = (xy'-y)\left(\frac{1+y'^2}{y''}\right)$

using ①

$x^2+y^2+\left(\frac{x^2+y^2}{-2}\right) = (xy'-y)\left(\frac{1+y'^2}{y''}\right)$

$\frac{x^2+y^2}{2} = (xy'-y)\left(\frac{1+y'^2}{y''}\right)$

$(x^2+y^2)y'' = 2(xy'-y)(1+y'^2)$

(iii) Find the diff eq of ellipses in standard form.

Ellipses in standard form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ —— (1)

Diff twice because two const $a, b$.

$$\frac{2x}{a^2}+\frac{2y}{b^2}y'=0$$

$$\frac{x}{a^2}+\frac{yy'}{b^2}=0 \quad \text{—— (2)}$$

Diff $\frac{1}{a^2}+\frac{yy''+y'^2}{b^2}=0$

×by $x$ $\Rightarrow$ $\frac{x}{a^2}+\frac{x(yy''+y'^2)}{b^2}=0$

$$\Rightarrow \frac{x}{a^2}=-\frac{x(yy''+y'^2)}{b^2} \quad \text{—— (3)}$$

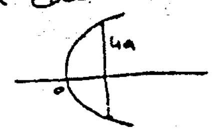
Put (3) in (2) $\Rightarrow$ $-\frac{x(yy''+y'^2)}{b^2}+\frac{yy'}{b^2}=0$

$$\Rightarrow -xyy''-xy'^2+yy'=0$$

$$xy'^2+xyy''-yy'=0$$

×————×

(iv) Find the diff eq of Parabolas each of which has a latus rectum $4a$ and whose axes are ∥ to $x$-axis.




$y^2=4ax$

$(y-k)^2=4a(x-h)$

∴ axis ∥ to $x$-axis.

Eq of given parabola is $(y-k)^2=4a(x-h)$ —— (1)

Diff twice because two const $h, k$.

$$2(y-k)y'=4a$$

$$(y-k)y'=2a \quad \text{—— (2)}$$

$$y'y'+(y-k)y''=0$$

$$y'^2+(y-k)y''=0$$

$$(y-k)=-\frac{y'^2}{y''} \quad \text{—— (3)}$$

Put (3) in (2)

$$\left[-\frac{y'^2}{y''}\right]y'=2a$$

$$-y'^3=2ay''$$

$$0=2ay''+y'^3$$

×————×

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(v) Find diff eq of Hyperbolas in standard form.

standard eq of Hyperbolas is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ —① Diff twice ∵ two const a, b

Diff $\frac{2x}{a^2}-\frac{2y}{b^2}y'=0$

$\frac{x}{a^2}-\frac{yy'}{b^2}=0$ —②

Diff $\frac{1}{a^2}-\frac{(yy''+y'^2)}{b^2}=0$

× by x ⇒ $\frac{x}{a^2}-\frac{x(yy''+y'^2)}{b^2}=0$ —

⇒ $\frac{x}{a^2}=\frac{x(yy''+y'^2)}{b^2}$ —③

Put ③ in ②

$\frac{x(yy''+y'^2)}{b^2}-\frac{yy'}{b^2}=0$

$xyy''+xy'^2-yy'=0$

vi) Find diff eq of conics which coincide with the axes of coordinates. Find diff eq of conics whose axes coincide with axes of coord.

$ax^2+by^2=1$ —① is Eq of conics whose axes coincide with axes of coord. Diff twice because two const a, b

Diff $2ax+2byy'=0$

$ax+byy'=0$ —②

Diff $a+b(yy''+y'^2)=0$ —③

Eliminating a, b from ①②,③

$$\begin{vmatrix} x^2 & y^2 & 1 \\ x & yy' & 0 \\ 1 & yy''+y'^2 & 0 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} x & yy' \\ 1 & yy''+y'^2 \end{vmatrix}=0$$

$\Rightarrow x(yy''+y'^2)-yy'=0$

(12)

(4) Solve the following initial value problems. (at one value of x)

(i) $\frac{dy}{dx} = -\frac{x}{y}$ , $y(3) = 4$

G. Sol is $x^2 + y^2 = c^2$

$3^2 + 4^2 = c^2 \Rightarrow \boxed{c = 5}$

$\therefore x^2 + y^2 = 25$ is req sol.

(ii) $\frac{dy}{dx} + y = 2x e^{-x}$, $y(-1) = e+3$

G. Sol is $y = (x^2 + c) e^{-x}$

$e+3 = (1+c) e^{-(-1)}$ $\quad \because y(-1) = e+3$

$\cancel{e} + 3 = \cancel{e} + ce \Rightarrow \boxed{c = \frac{3}{e}}$

$\therefore y = (x^2 + \frac{3}{e}) e^{-x}$ is Particular Sol

(iii) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 12 = 0$ , $y(0) = -2$, $y'(0) = 6$

G. Sol is $y = A e^{4x} + B e^{-3x}$ —— (I)

$\therefore (-2) = A e^0 + B e^0$ $\quad \because y(0) = -2$

$-2 = A + B$ —— (II)

diff (I) $y' = 4A e^{4x} - 3B e^{-3x}$

$6 = 4A e^0 - 3B e^0$ $\quad \because y'(0) = 6$

$6 = 4A - 3B$ —— (III)

solving (II) & (III)

× (II) by 4 · $-8 = \cancel{4A} + 4B$

$\underline{\ \ 6 = \cancel{4A} - 3B}$
$\phantom{6}\ - \quad\ \ - \quad +$

$-14 = 7B$ $\quad \boxed{B = -2}$

using (II) $\boxed{0 = A}$. $\therefore y = -2 e^{-3x}$ is P. Sol.

(iv) $x \frac{dy}{dx} + 2y = 4x^2$ $\quad y(1) = 2$

G. Sol is $y = x^2 + \frac{c}{x^2}$

$2 = 1^2 + \frac{c}{1^2}$ $\quad \because y(1) = 2$

$\boxed{1 = c}$

$\therefore$ P. Sol is $y = x^2 + \frac{1}{x^2}$

(13)

(v) $x^3\frac{d^3y}{dx^3} - 3x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} - 6y = 0$ , $y(2)=0$ , $y'(2)=2$, $y''(2)=6$

G.Sol is $y = c_1x + c_2x^2 + c_3x^3$ —(I)

$y' = c_1 + c_2(2x) + c_3(3x^2)$ —(II)

$y'' = 2c_2 + c_3 6x$ —(III)

from (I) $0 = 2c_1 + 4c_2 + 8c_3$ (IV) $\because y(2)=0$

from (II) $2 = c_1 + 4c_2 + 12c_3$ (V) $\because y'(2)=2$

from (III) $6 = 2c_2 + 12c_3$ (VI) $\because y''(2)=6$

from (IV) $0 = c_1 + 2c_2 + 4c_3$

from (V) $2 = c_1 + 4c_2 + 12c_3$ subtracting

$-2 = -2c_2 - 8c_3$

from (VI) $6 = 2c_2 + 12c_3$ Adding

$4 = 0 + 4c_3$

$\boxed{1 = c_3}$

from (VI) $\boxed{-3 = c_2}$

from (V) $\boxed{2 = c_1}$

$\therefore$ P.Sol is $y = 2x - 3x^2 + x^3$

5 (i) Solve boundary value Problem (values of x are more than one)

$\frac{d^2y}{dx^2} + y = 0$ , $y(0) = 1$ $y'(\frac{\pi}{2}) = -1$

G.Sol is $y = c_1 \sin x + c_2 \cos x$ —(I)

$y' = c_1 \cos x - c_2 \sin x$ —(II)

from (I) $\boxed{1 = c_2}$ $\because y(0)=1$

from (II) $\boxed{-1 = -c_2}$ $\because y'(\frac{\pi}{2}) = -1$

$y = c_1 \sin x + \cos x$ is P.Sol.

(ii) $\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$, $y(0) = 0$ $y(1) = 1$

$y = c_1 e^{x} + c_2 e^{3x}$ is the G. Sol — (I)

$0 = c_1 e^{0} + c_2 e^{0}$ $\because y(0) = 0$

$0 = c_1 + c_2$ — (II) $\Rightarrow c_1 = -c_2$

$1 = c_1 e + c_2 e^{3}$ $\because y(1) = 1$

$\frac{1}{e} = c_1 + c_2 e^{2}$ — (III) $\div$ by $e$

Subtract (III) from (II)

$$c_1 + c_2 = 0$$
$$c_1 + c_2 e^2 = \frac{1}{e}$$
$$c_2 - c_2 e^2 = -\frac{1}{e}$$

$c_2(1-e^2) = -\frac{1}{e}$ $\Rightarrow c_2 = \frac{-1/e}{1-e^2}$

$\Rightarrow c_2 = \frac{-1}{e(1-e^2)}$

using (II) $\therefore$ $c_1 = \frac{1}{e(1-e^2)}$

$$y = \frac{1}{e(1-e^2)} e^{x} + \left(\frac{-1}{e(1-e^2)}\right) e^{3x}$$

$$= \frac{1}{e(1-e^2)}\left[e^{x} - e^{3x}\right]$$ Ans.

If $c_1$ or $c_2$ has two different values as $c_1 = -1$ & $c_1 = 2$ then we can not determine $c_2$, hence No Solution exist.

See Example 7.

9.2-01

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## Differential Eqs of First order and First Degree..

An ordinary differential eq of first order and first degree can be expressed as $\frac{dy}{dx} = f(x,y)$

or $M(x,y)\,dx + N(x,y)\,dy = 0$

The General sol of such an eq will contain only one arbitrary const.

We discuss these types of diffl eqs of first degree & first order.

1) Separable Variables Eqs
2) Homogeneous Eqs
3) Non Homogenous Eqs.
4) Exact Eqs
5) Non Exact. Eqs
6) Linear Eqs
7) Bernoulli Eqs

**Type 1** Separable Eq. is a diffl eq of the form $M(x)dx + N(y)dy = 0$

where $M(x)$ is fn of $x$ alone and $N(y)$ is fn of $y$ alone.

**To Solve** we separate variables & integrate.

**Note** Constant of integration 'c' can be replaced by $\log c$, $e^c$, $\tan^{-1}c$ etc whichever is suitable for simplification.

**Note** Since '$\ln x$' when $x$ is negative is not defined so it is better to write $\ln|x|$ ie modulus when there is possibility of a -ve number.

Ex 9.2

Solve

(1) $\frac{dy}{dx} = \frac{x^2}{y(1+x^3)}$

$y\,dy = \frac{x^2}{1+x^3}\,dx$

$\int y\,dy = \frac{1}{3}\int\frac{3x^2}{1+x^3}\,dx$

$\frac{y^2}{2} = \frac{1}{3}\ln(1+x^3)+c$

$\frac{3y^2}{2} = \ln(1+x^3)+3c$

$3y^2 = 2\ln(1+x^3)+6c$

$3y^2 = 2\ln(1+x^3)+c'$

(2) $\frac{dy}{dx} + y^2\sin x = 0$

$\frac{dy}{dx} = -y^2\sin x$

$\int\frac{dy}{y^2} = -\int\sin x\,dx$

$\frac{y^{-1}}{-1} = -(-\cos x)+c$

$-\frac{1}{y} = \cos x + c$

(3) $\frac{dy}{dx} = 1+x+y^2+xy^2$

$\frac{dy}{dx} = (1+x)+y^2(1+x)$

$\frac{dy}{dx} = (1+x)(1+y^2)$

$\int\frac{dy}{1+y^2} = \int(1+x)\,dx$

$\tan^{-1}y = x+\frac{x^2}{2}+c$

$2\tan^{-1}y = 2x+x^2+c'$

(5) $\frac{dy}{dx} = 2x^2+y-x^2y+xy-2x-2$

$= 2x^2-2x-2+y-x^2y+xy$

$= 2(x^2-x-1)-y(-1+x^2-x)$

$\frac{dy}{dx} = (x^2-x-1)(2-y)$

$\int\frac{dy}{2-y} = \int(x^2-x-1)\,dx$

$-\int\frac{-dy}{2-y} = \int(x^2-x-1)\,dx$

$-\ln|2-y| = \frac{x^3}{3}-\frac{x^2}{2}-x+c$

$-\ln|2-y| = \frac{2x^3-3x^2-6x+6c}{6}$

$-6\ln|2-y| = 2x^3-3x^2-6x+6c$

$\ln|2-y|^{-6} = (2x^3-3x^2-6x+6c)\ln e$

$\ln|2-y|^{-6} = \ln e^{2x^3-3x^2-6x+6c}$

$|2-y|^{-6} = e^{2x^3-3x^2-6x}\cdot e^{6c}$

$|2-y|^{-6} = c_1 e^{2x^3-3x^2-6x}$

(4) $(xy+2x+y+2)dx+(x^2+2x)dy=0$

$[x(y+2)+(y+2)]dx + x(x+2)dy = 0$

$[(y+2)(x+1)]dx + x(x+2)dy = 0$

$\div$ by $x(x+2)(y+2)$

$\frac{x+1}{x(x+2)}\,dx + \frac{1}{y+2}\,dy = 0$

$\int\frac{x+1}{x^2+2x}\,dx + \int\frac{dy}{y+2} = 0$

$\frac{1}{2}\int\frac{2x+2}{x^2+2x}\,dx + \int\frac{dy}{y+2} = 0$

$\ln(y+2) = -\frac{1}{2}\ln(x^2+2x)+\ln c$

$y+2 = \frac{c}{\sqrt{x^2+2x}}$

⑥ $\operatorname{Cosec} y\, dx + \sec x\, dy = 0$

$\div$ by $\operatorname{Cosec} y \sec x$

$$\Rightarrow \frac{1}{\sec x}dx + \frac{dy}{\operatorname{Cosec} y} = 0$$

$$\Rightarrow \int \cos x\, dx + \int \sin y\, dy = \int 0\, dx$$

$$\Rightarrow \sin x - \cos y = c \quad \text{general Sol}$$

⑦ $y(1+x)dx + x(1+y)dy = 0$

$\div$ by $xy$

$$\Rightarrow \frac{(1+x)}{x}dx + \frac{(1+y)}{y}dy = 0$$

$$\Rightarrow \int\left(\frac{1}{x}+1\right)dx + \int\left(\frac{1}{y}+1\right)dy = \int 0\, dx$$

$$\Rightarrow \ln x + x + \ln y + y = c$$

$$\Rightarrow x + y + \ln(xy) = c$$

⑨ $\dfrac{dy}{dx} + \sqrt{\dfrac{1-y^2}{1-x^2}} = 0$

$$\frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}} \quad \text{if } |x|<1,\ |y|<1$$

$$\text{or} \int \frac{dy}{\sqrt{1-y^2}} = -\int \frac{dx}{\sqrt{1-x^2}}$$

$$\sin^{-1} y = -\sin^{-1} x + c$$

$$y = \sin(c - \sin^{-1} x) \text{ is G.Sol.}$$

$$\frac{dy}{dx} + \sqrt{\frac{y^2-1}{x^2-1}} = 0 \quad \text{if } |x|>1,\ |y|>1$$

$$\frac{dy}{dx} = -\sqrt{\frac{y^2-1}{x^2-1}}$$

$$\int \frac{dy}{\sqrt{y^2-1}} = -\int \frac{dx}{\sqrt{x^2-1}}$$

$$\cosh^{-1} y = -\cosh^{-1} x + c$$

$$y = \cosh(c - \cosh^{-1} x)$$

⑧ $y\sqrt{1+x^2}\,dx + x\sqrt{1+y^2}\,dy = 0$

$\div$ by $xy$

$$\Rightarrow \int \frac{\sqrt{1+x^2}}{x}dx + \int \frac{\sqrt{1+y^2}}{y}dy = \int 0\, dx$$

Put $\sqrt{1+x^2} = t$

$1+x^2 = t^2$

$2x\,dx = 2t\,dt$

$x\,dx = t\,dt$

Put $\sqrt{1+y^2} = z$

$1+y^2 = z^2$

$2y\,dy = 2z\,dz$

$y\,dy = z\,dz$

Therefore

$$\Rightarrow \int \frac{\sqrt{1+x^2}}{x^2}x\,dx + \int \frac{\sqrt{1+y^2}}{y^2}y\,dy = \int 0\, dx$$

$$\Rightarrow \int \frac{t \cdot t\,dt}{t^2-1} + \int \frac{z \cdot z\,dz}{z^2-1} = c$$

$$\Rightarrow \int\left(\frac{t^2-1+1}{t^2-1}\right)dt + \int \frac{z^2-1+1}{z^2-1}dz = c$$

$$\Rightarrow \int\left(1 + \frac{1}{t^2-1}\right)dt + \int\left(1 + \frac{1}{z^2-1}\right)dz = c$$

$$\Rightarrow t + \frac{1}{2}\ln\left(\frac{t-1}{t+1}\right) + z + \frac{1}{2}\ln\left(\frac{z-1}{z+1}\right) = c$$

$$\Rightarrow \sqrt{1+x^2} + \frac{1}{2}\ln\left(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right) + \sqrt{1+y^2} + \frac{1}{2}\ln\left(\frac{\sqrt{1+y^2}-1}{\sqrt{1+y^2}+1}\right) = c$$

⑩ $(e^x+1)y\,dy = (y+1)e^x\,dx$

$\div$ by $(e^x+1)(y+1)$

$$\Rightarrow \int \frac{y\,dy}{y+1} = \int \frac{e^x\,dx}{e^x+1}$$

$$\Rightarrow \int\left(\frac{y+1-1}{y+1}\right)dy = \int \frac{e^x}{e^x+1}dx$$

$$\Rightarrow \int\left(1 - \frac{1}{y+1}\right)dy = \int \frac{e^x\,dx}{e^x+1}$$

$$\Rightarrow y - \ln(y+1) = \ln(e^x+1) + \ln c$$

$$\Rightarrow y = \ln(y+1) + \ln(e^x+1) + \ln c$$

$$\Rightarrow y = \ln\{(y+1)(e^x+1)c\}$$

$$\Rightarrow e^y = c(y+1)(e^x+1)$$

⑪ $\frac{dy}{dx} = \frac{y^3+2y}{x^2+3x}$

$\frac{dy}{y^3+2y} = \frac{dx}{x^2+3x}$

By Partial Fractions from ① & ⑪

$\left[\frac{1}{2y} - \frac{y}{2(y^2+2)}\right]dy = \left[\frac{1}{3x} - \frac{1}{3(x+3)}\right]dx$

$\int\frac{dy}{2y} - \int\frac{(2y)\,dy}{4(y^2+2)} = \int\frac{dx}{3x} - \int\frac{dx}{3(x+3)}$

$\frac{1}{2}\ln y - \frac{1}{4}\ln(y^2+2) = \frac{1}{3}\ln x - \frac{1}{3}\ln(x+3) + \ln c$

$\ln y^{\frac{1}{2}} - \ln(y^2+2)^{\frac{1}{4}} = \ln x^{\frac{1}{3}} - \ln(x+3)^{\frac{1}{3}} + \ln c^{\frac{1}{3}}$

$\ln\left[\frac{y^{\frac{1}{2}}}{(y^2+2)^{\frac{1}{4}}}\right] = \ln\left[\frac{cx}{(x+3)}\right]^{\frac{1}{3}}$

$\frac{y^{\frac{1}{2}}}{(y^2+2)^{\frac{1}{4}}} = \left(\frac{cx}{x+3}\right)^{\frac{1}{3}}$ is G. Sol.

Partial Fractions

$\frac{1}{y(y^2+2)} = \frac{A}{y} + \frac{By+C}{y^2+2}$

$1 = A(y^2+2) + (By+C)y$

Put $y=0 \Rightarrow 1 = A(2) \Rightarrow \boxed{A=\frac{1}{2}}$

comparing coeff of $y^2 \rightarrow 0 = A+B$

$0 = \frac{1}{2}+B \quad \boxed{B=-\frac{1}{2}}$

comparing coeff of $y \Rightarrow \boxed{C=0}$

Thus $\frac{1}{y(y^2+2)} = \frac{1}{2y} - \frac{1(y)+0}{2(y^2+2)}$

$\frac{1}{y(y^2+2)} = \frac{1}{2y} - \frac{y}{2(y^2+2)}$ — ①

Now $\frac{1}{x(x+3)} = \frac{A}{x} + \frac{B}{x+3}$

$1 = A(x+3) + Bx$

Put $x=0 \Rightarrow \boxed{A=\frac{1}{3}}$

Put $x+3=0 \Rightarrow \boxed{B=-\frac{1}{3}}$

$\therefore \frac{1}{x(x+3)} = \frac{1}{3x} - \frac{1}{3(x+3)}$ — ⑪

⑫ $(\sin x + \cos x)dy + (\cos x - \sin x)dx = 0$

$\div$ by $(\sin x + \cos x)$

$\int dy + \int\frac{(\cos x - \sin x)}{(\sin x + \cos x)}dx = \int 0\,dx$

$y + \ln(\sin x + \cos x) = c$

$y\ln e + \ln(\sin x + \cos x) = c\ln e$

$\ln e^y + \ln(\sin x + \cos x) = \ln e^c$

$\ln\left[e^y(\sin x + \cos x)\right] = \ln e^c$

$e^y(\sin x + \cos x) = c'$

$e^y = \frac{c'}{\sin x + \cos x}$ G. Sol.

⑬ $(2x\cos y)dx + x^2(\sec y - \sin y)dy = 0$

$\div$ by $x^2\cos y$

$\frac{2x\cos y}{x^2\cos y}dx + \frac{x^2(\sec y - \sin y)}{x^2\cos y}dy = 0$

$\int\frac{2dx}{x} + \int\left(\frac{\sec y}{\cos y} - \frac{\sin y}{\cos y}\right)dy = \int 0$

$2\int\frac{dx}{x} + \int(\sec^2 y - \tan y)dy = \int 0\,dx$

$2\ln x + \tan y - \ln\sec y = c$

$2\ln x = \ln\sec y - \tan c + c$

G. Sol

(13) 2nd $e^{x}\left(1+\frac{dy}{dx}\right)=xe^{-y}$

$$1+\frac{dy}{dx} = xe^{-y}e^{-x}$$

$$1+\frac{dy}{dx} = xe^{-(x+y)}$$

Not separable

So Put $z=x+y$

$$\frac{dz}{dx}=1+\frac{dy}{dx}$$

$$\therefore \quad \frac{dz}{dx} = xe^{-z}$$

$$\int e^{z}\,dz = \int x\,dx$$

$$e^{z} = \frac{x^2}{2}+c$$

$$e^{x+y} = \frac{x^2}{2}+c$$

$$\ln e^{x+y} = \ln\left(\frac{x^2}{2}+c\right)$$

$$(x+y)\ln e = \ln\left(\frac{x^2}{2}+c\right)$$

$$(x+y)\,1 = \ln\left(\frac{x^2}{2}+c\right)$$

$$y = \ln\left(\frac{x^2}{2}+c\right)-x \quad \text{G.Sol.}$$

(14) $xe^{x^2+y}\,dx = y\,dy$

$$xe^{x^2}e^{y}\,dx = y\,dy$$

$$xe^{x^2}\,dx = ye^{-y}\,dy$$

$$\frac{1}{2}\int e^{x^2}(2x)\,dx = \int ye^{-y}\,dy \quad \text{IBP}$$

$$\because \begin{cases} e^{x^2}=t & e^{x^2}2x\,dx=dt \\ \int e^{x^2}2x\,dx=\int dt=t=e^{x^2} \end{cases}$$

$$\text{So } \frac{1}{2}e^{x^2} = y\frac{e^{-y}}{-1}-\int 1\cdot\frac{e^{-y}}{-1}\,dy$$

$$= -ye^{-y}+\int e^{-y}\,dy$$

$$= -ye^{-y}+\frac{e^{-y}}{-1}+c$$

$$\frac{1}{2}e^{x^2} = -ye^{-y}-e^{-y}+c$$

$$e^{x^2} = -2e^{-y}(y+1)+2c$$

$$e^{x^2} = -2e^{-y}(y+1)+c' \quad \text{G.Sol.}$$

(16) Solve the initial value problems.

$$2(y-1)\,dy = (3x^2+4x+2)\,dx \qquad y(0)=-1$$

$$2\int(y-1)\,dy = \int(3x^2+4x+2)\,dx$$

$$2\left(\frac{y^2}{2}-y\right) = 3\frac{x^3}{3}+4\frac{x^2}{2}+2x+c$$

$$\cancel{2}\left(\frac{y^2-2y}{\cancel{2}}\right) = x^3+2x^2+2x+c$$

$$\because y(0)=-1$$

$$\therefore (-1)^2-2(-1) = 0+0+0+c$$

$$\boxed{3 = c}$$

$$\text{Hence } y^2-2y = x^3+2x^2+2x+3$$

$$+1-1 \quad y^2-2y+1 = x^3+2x^2+2x+3+1$$

$$(y-1)^2 = x^3+2x^2+2x+4$$

$$y-1 = \pm\sqrt{x^3+2x^2+2x+4}$$

$$y = 1\pm\sqrt{x^3+2x^2+2x+4}$$

$y(0)=-1$ doesnot satisfy $y=1+\sqrt{x^3+2x^2+2x+4}$

$$\because -1 = 1+\sqrt{0+0+0+4}$$

$$-1 = 1+2 \quad \text{Impossible}$$

$y(0)=-1$ satisfy $y=1-\sqrt{x^3+2x^2+2x+4}$

$$-1 = 1-\sqrt{0+0+0+4}$$

$$-1 = 1-2 \quad \text{true}$$

So $y=1-\sqrt{x^3+2x^2+2x+4}$ is Sol.

Q (17) $(3x+8)(y^2+4)dx - 4y(x^2+5x+6)dy = 0$, $y(1) = 2$

$\div$ by $(y^2+4)(x^2+5x+6)$

$$\Rightarrow \frac{3x+8}{x^2+5x+6}dx - \frac{4y}{y^2+4}dy = 0$$

By Partial Fractions

$$\frac{3x+8}{x^2+5x+6}$$

$$\therefore \frac{3x+8}{(x+2)(x+3)} = \frac{A}{(x+2)} + \frac{B}{x+3} \quad \text{— (I)}$$

$$3x+8 = A(x+3) + B(x+2) \quad \text{— (II)}$$

Put $x+3=0 \Rightarrow x=-3$
$\Rightarrow \boxed{B=1}$

Put $x+2=0 \Rightarrow x=-2$
$\Rightarrow \boxed{A=2}$

$$\therefore \frac{3x+8}{(x+2)(x+3)} = \frac{2}{(x+2)} + \frac{1}{x+3}$$

$$\Rightarrow \left(\frac{2}{x+2} + \frac{1}{x+3}\right)dx - \frac{4y}{y^2+4}dy = 0$$

$$\Rightarrow \int\frac{2}{x+2}dx + \int\frac{dx}{x+3} - 2\int\frac{2y}{y^2+4}dy = \int 0\,dx$$

$$\Rightarrow 2\ln(x+2) + \ln(x+3) - 2\ln(y^2+4) = \ln c$$

$$\ln\left[\frac{(x+2)^2(x+3)}{(y^2+4)^2}\right] = \ln c$$

antilog $$\frac{(x+2)^2(x+3)}{(y^2+4)^2} = c$$

$\because y(1)=2 \Rightarrow \left.\begin{matrix} y=2 \\ x=1 \end{matrix}\right) \Rightarrow c = \dfrac{(1+2)^2(1+3)}{(2^2+4)^2}$

$$c = \frac{36}{64} = \frac{9}{16}$$

$$\therefore \frac{(x+2)^2(x+3)}{(y^2+4)^2} = \frac{9}{16}$$

$$16(x+2)^2(x+3) = 9(y^2+4)^2 \quad \text{Ans.}$$



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(18) $(1+2y^2)dy = y\cos x\,dx$, $y(0)=1$

$\div$ by $y$.

$\Rightarrow \left(\frac{1+2y^2}{y}\right)dy = \cos x\,dx$

$\Rightarrow \int\left(\frac{1}{y}+2y\right)dy = \int\cos x\,dx$

$\Rightarrow \ln y + \cancel{2}\frac{y^2}{\cancel{2}} = \sin x + C$

$\because y(0)=1$

$\therefore \ln 1 + 1 = 0 + C$

$\boxed{1 = C}$

$\Rightarrow \therefore \ln y + y^2 = \sin x + 1$

(19) $8\cos^2 y\,dx + \operatorname{cosec}^2 x\,dy = 0$, $y\left(\frac{\pi}{12}\right) = \frac{\pi}{4}$

$\div$ by $\cos^2 y\operatorname{cosec}^2 x$

$\Rightarrow \frac{8}{\operatorname{cosec}^2 x}dx + \frac{1}{\cos^2 y}dy = 0$

$\Rightarrow \int 8\sin^2 x\,dx + \int\sec^2 y\,dy = \int 0\,dx$

$\Rightarrow 4\int 2\sin^2 x\,dx + \tan y = C$

$\Rightarrow 4\int(1-\cos 2x)dx + \tan y = C$

$\Rightarrow 4\left(x - \frac{\sin 2x}{2}\right) + \tan y = C$

$\Rightarrow 4x - 2\sin 2x + \tan y = C$

$\Rightarrow \tan y = -4x + 2\sin 2x + C$

$\because y\left(\frac{\pi}{12}\right) = \frac{\pi}{4}$

$\therefore \tan\left(\frac{\pi}{4}\right) = -4\left(\frac{\pi}{12}\right) + 2\sin 2\left(\frac{\pi}{12}\right) + C$

$1 = -\frac{\pi}{3} + 2\sin\frac{\pi}{6} + C$

$1 = -\frac{\pi}{3} + 2\left(\frac{1}{2}\right) + C$

$\cancel{1} - \cancel{1} + \frac{\pi}{3} = C \Rightarrow \boxed{C = \frac{\pi}{3}}$

$\Rightarrow \therefore \tan y = -4x + 2\sin 2x + \frac{\pi}{3}$ Ans

(20) $\frac{dy}{dx} = \frac{x(x^2+1)}{4y^3}$, $y(0) = \frac{-1}{\sqrt{2}}$

$\Rightarrow 4y^3\,dy = x(x^2+1)\,dx$

$\Rightarrow 4\int y^3\,dy = \int(x^3+x)\,dx$

$\Rightarrow \cancel{4}\frac{y^4}{\cancel{4}} = \frac{x^4}{4} + \frac{x^2}{2} + C$

$\because y(0) = -\frac{1}{\sqrt{2}}$

$\therefore \left(-\frac{1}{\sqrt{2}}\right)^4 = 0 + 0 + C$

$\boxed{\frac{1}{4} = C}$

$\Rightarrow \therefore y^4 = \frac{x^4}{4} + \frac{x^2}{2} + \frac{1}{4}$

$\Rightarrow 4y^4 = x^4 + 2x^2 + 1$

$\Rightarrow 4y^4 = (x^2+1)^2$

$\Rightarrow 2y^2 = x^2 + 1$

$\Rightarrow y^2 = \frac{x^2+1}{2}$

$\Rightarrow y = \pm\sqrt{\frac{x^2+1}{2}}$

$\Rightarrow y = -\sqrt{\frac{x^2+1}{2}}$ Ans.

Note: $y = +\sqrt{\frac{x^2+1}{2}}$ is not satisfied by $y(0) = -\frac{1}{\sqrt{2}}$. So leave it.

## Homogeneous Diff Eq. (H.DE)

A differential eq. of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ is said to be homogeneous diff eq. if both fns $f(x,y)$ & $g(x,y)$ are homogeneous of same degree.

**Homogeneous Fn:-**

A function $f(x,y)$ is said to be homogeneous of degree 'n', if it can be written as $f(tx, ty) = t^n f(x,y)$

e.g. $f(x,y) = \sqrt{xy}$
$f(tx,ty) = \sqrt{tx\,ty} = t\sqrt{xy}$
$f(x,y)$ is homogeneous fn of degree '1'

**To Solve** Put $y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

then by method of separable variable we solve.

## Ex 9.3

(1) $(x-y)dx + (x+y)dy = 0$

$xdx - ydx + xdy + ydy = 0$
Not separable.

$$(x+y)\,dy = -(x-y)\,dx$$

$$\frac{dy}{dx} = \frac{y-x}{x+y} \quad \text{H.DE} \quad (i)$$

Put $\quad y = vx \quad (ii)$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad (iii)$$

using (ii) (iii) in (i)

$$v + x\frac{dv}{dx} = \frac{vx - x}{x + vx}$$

$$x\frac{dv}{dx} = \frac{\cancel{x}(v-1)}{\cancel{x}(1+v)} - v$$

$$= \frac{\cancel{v} - 1 - \cancel{v} - v^2}{1+v}$$

$$x\frac{dv}{dx} = -\frac{(v^2+1)}{1+v}$$

$$\int \frac{v+1}{v^2+1}\,dv = -\int \frac{dx}{x}$$

$$\frac{1}{2}\int \frac{2v\,dv}{v^2+1} + \int \frac{dv}{v^2+1} = -\int \frac{dx}{x}$$

$$\frac{1}{2}\ln(v^2+1) + \tan^{-1} v = -\ln x + c$$

$$\ln(v^2+1)^{\frac{1}{2}} + \tan^{-1} v + \ln x = c$$

$$\ln\sqrt{\frac{y^2}{x^2}+1} + \tan^{-1}\left(\frac{y}{x}\right) + \ln x = c$$

$$\ln\sqrt{y^2+x^2} - \cancel{\ln\sqrt{x^2}} + \tan^{-1}\left(\frac{y}{x}\right) + \cancel{\ln x} = c$$

$$\ln\sqrt{y^2+x^2} + \tan^{-1}\left(\frac{y}{x}\right) = c$$